# A Flow Based Pruning Scheme For Enumerative Equitable Coloring Algorithms

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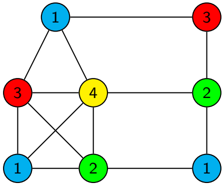
**The Problem:**

Given an undirected graph $G = (V, E)$,
the vertex coloring problem (CP) asks for

- the minimal $k \in \mathbb{Z}_+$,
- such there is

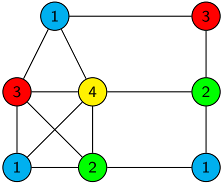
$$f : V \to \{1, \ldots, j\} : v \mapsto f(v),$$

- with $f(v) \neq f(w)$ for all $vw \in E$.

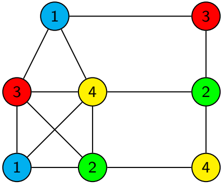



**Applications:**

- Assigning workers (colors) to conflicting jobs.
- Assigning machines (colors) to conflicting tasks.




vs.




**Solution:** Equitable vertex coloring (ECP) – An assignment $f$ where

- the sizes of the color classes may only differ by one,
- $\Big| |\{v \mid f(v) = j\}| - |\{v \mid f(v) = i\}| \Big| = 1 \forall\, i, j = 1, \dots k$, i.e.,
- each worker has at most one more task than any other worker.

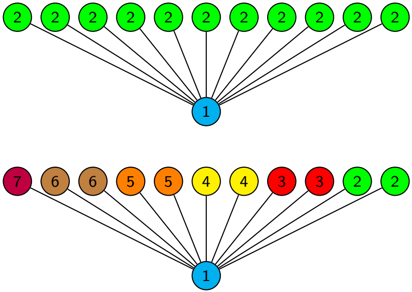
- The chromatic number $\chi(G)$ is the min. $k \in \mathbb{N}$ for which CP is solvable. The equitable chromatic number $\chi_e(G)$ denotes the same for ECP.

# Example: ECP

- The chromatic number $\chi(G)$ is the min. $k \in \mathbb{N}$ for which CP is solvable. The equitable chromatic number $\chi_e(G)$ denotes the same for ECP.

**Observation 1**

*The difference between $\chi(G)$ and $\chi_e(G)$ can be arbitrarily large.*




- I.e., consider a star with $k$ nodes: $\chi(S) = 2$ and $\chi_e(S) = \lceil \frac{k-1}{2} \rceil + 1$.

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**Observation 1**

*The difference between $\chi(G)$ and $\chi_e(G)$ can be arbitrarily large.*

- If the number of colors is fixed, so is the amount and the sizes of the color classes:

**Observation 2**

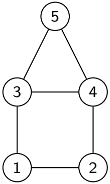
*Let $k \in \mathbb{N}$, $n = |V|$ and $p \equiv n \bmod k$. If $G$ admits an equitable coloring with $k$ colors, then there are $p$ color classes of size $\lceil \frac{n}{k} \rceil$ and $k - p$ color classes of size $\lfloor \frac{n}{k} \rfloor$.*

**DSATUR:** enumerate all possible colorings in a treelike structure.
Branch and Bound where possible (see [3, 4]).

```
Data: Graph G = (V, E) with a partial coloring Π_p
while T ≠ ∅;                      // T - the set of all partial colorings (leafs)
do
    Select Π_p ∈ T;                          // Select partial coloring (leaf)
    T ← T \ {Π_p};                                 // Remove this coloring
    if U(Π_p) := ∅;                  // Check whether the coloring is complete
    then
        Evaluate Π_p;                              // Check equitable coloring
    end
    /* V* as the uncolored vertices with max. dat. degree, choose one.    */
    Choose v ∈ V* := { v ∈ U(Π_p) | v = argmax { ρ_{Π_p}(v) } };
    for i ∈ F_{Π_p}(v);                 // For all free colors of this vertex
    do
        if not prune Π_p + < v, i >;                   // Check for pruning
        then
            T ← T ∪ {Π_p + < v, i >};    // Extend the coloring, add it to T.
        end
    end
end
```

# Example: DSATUR

**Example:** Lets do some (three-) coloring:
Start with the empty (partial) coloring as the single leaf.




The coloring tree, i.e., the list of explored partial colorings.

The current graph coloring.

- Only minor difference to DSATUR for standard graph coloring.
- Node/color selection for extending partial colorings is not trivial.
- Branching, bounding and pruning was not shown.

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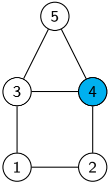


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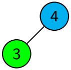


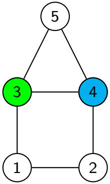


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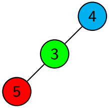


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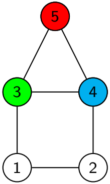


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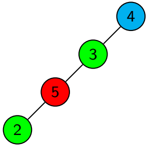


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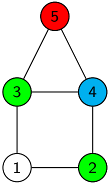


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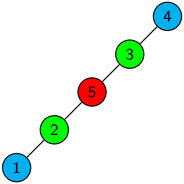


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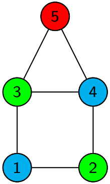


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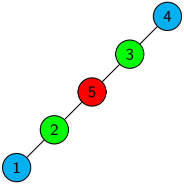


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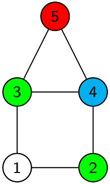


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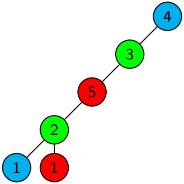


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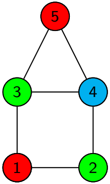


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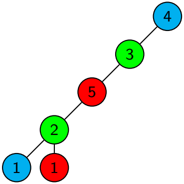


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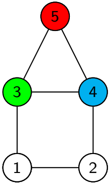


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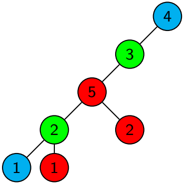


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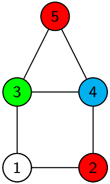


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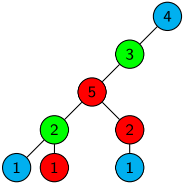


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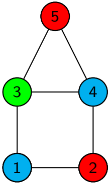


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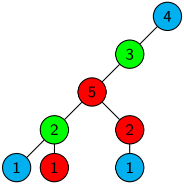


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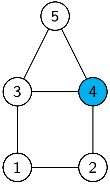


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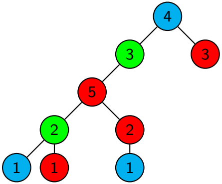


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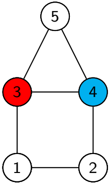


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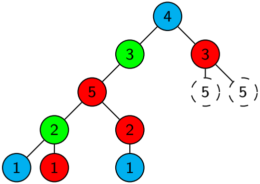


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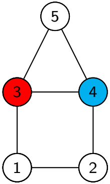


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**Notation:** given a partial coloring $\Pi_p$.

- $M(\Pi_p) := \max\{|C_i| \mid i = 1, \ldots, n\}$ as the size of the largest color class.
- $T(\Pi_p) := \{i \in \{1, \ldots, n\} \mid |C_i| = M(\Pi_p)\}$ as the indices of the largest color classes, and its cardinality $t(\Pi_p) := |T(\Pi_p)|$.

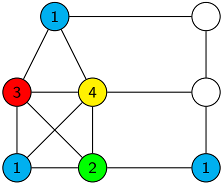
**Theorem 1 (Méndez-Díaz et al. [4])**

*If $\Pi_p$ can be extended to an equitable $k$-coloring, then*

$$\begin{aligned} n \geq & (M(\Pi_p) - 1)(k - t(\Pi_p)) + M(\Pi_p)t(\Pi_p) \\ = & (M(\Pi_p) - 1)k + t(\Pi_p). \end{aligned}$$

*Or, given a lower bound $\underline{k}$ for $\chi_{eq}(G)$, it is* $n \geq (M(\Pi_p) - 1)\max\{\underline{k}, k\} + t(\Pi_p)$.

**Example:** Consider the partial coloring



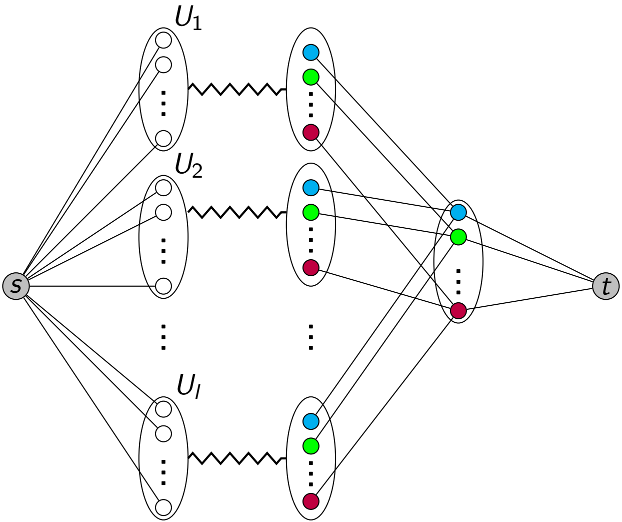

**Question:** Is it extendable?

- The biggest color class is blue, containing three nodes.
- Three color classes do not have enough nodes (for the coloring to be quitable).
- They need to contain at least two nodes each.
- But only two nodes are still uncolored.

# Contribution: Extended Pruning Rules

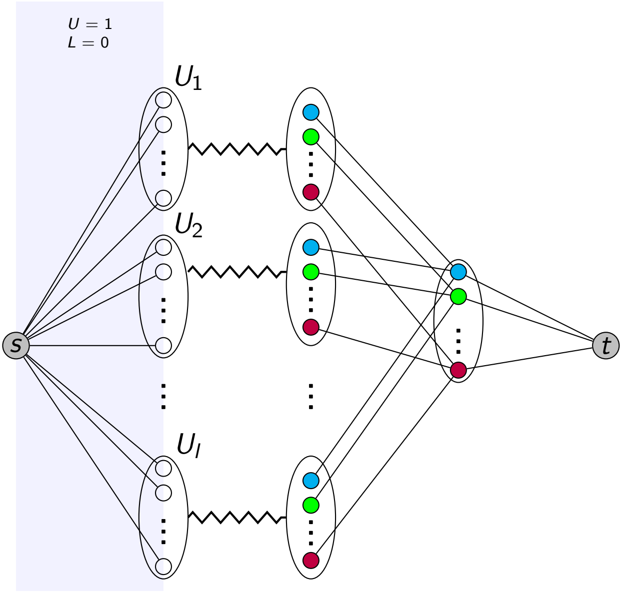
**Idea:** Deciding whether a partial coloring ($\Pi_p$) is extendable to a $k_0$ equitable coloring.

- Model extendability by some $s - t$ network.
- If there is no $s - t$ flow of a certain value, the partial coloring is not extendable to a $k_0$-coloring.
- If there is a $s - t$ flow of a certain value, the flow can yield an extension to a $k_0$-coloring.
- Similar to a network model presented by de Werra [1, 2].
- Theorem 1 states a special case implying the condition.
- Let $U(\Pi_p) = \biguplus U_i$.
  Define the directed network $N(G, \Pi_p, k_0) := (V_N, A_N)$.
  Visualization: next slide.



$U_1$
$U_2$
$U_l$
$s$
$t$

**Capacities:**



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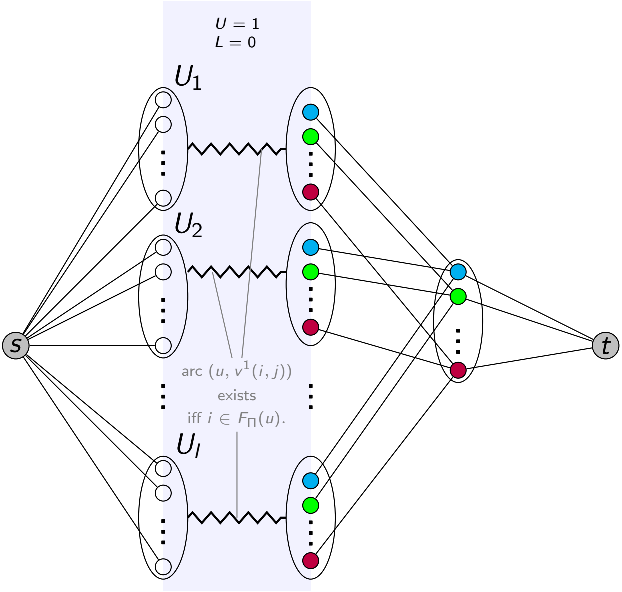

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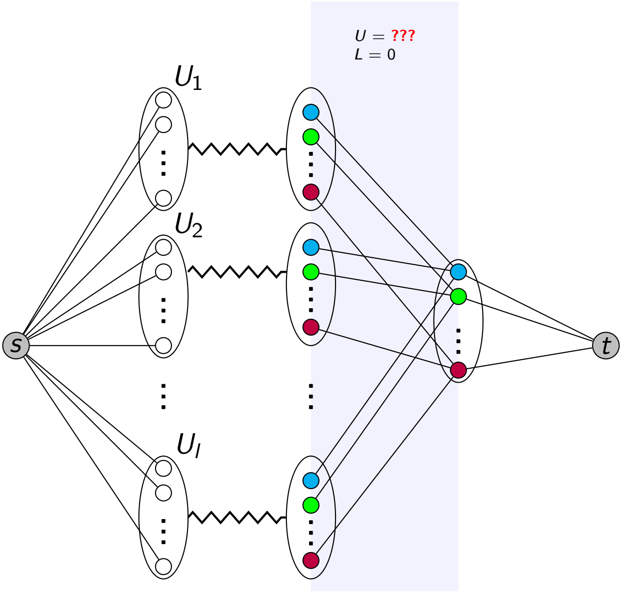
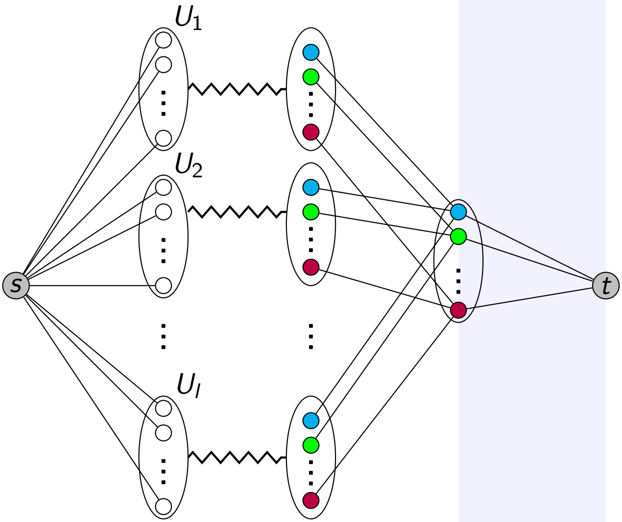


**Capacities:**

$U = \lceil \frac{n}{k_0} \rceil - |C_i|$
$L = \lfloor \frac{n}{k_0} \rfloor - |C_i|$



# Conditions for Extendability

Given a partial $k$-coloring $\Pi_p$.

**Theorem 2**

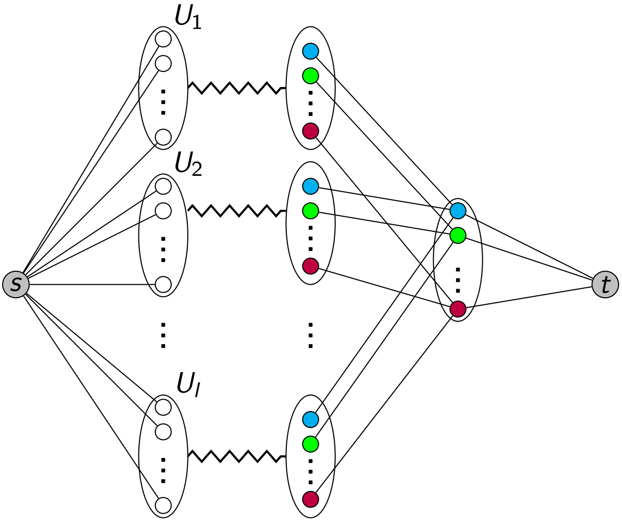
*Assume $k_0 \geq k$ and $U(\Pi_p) = \biguplus U_i$.*

*If $\Pi_p$ can be extended to an equitable $k_0$-coloring, then the network $N(G, \Pi_p, k_0)$ has an admissible flow of value $|U(\Pi_p)|$.*

Finding a "good"' decomposition of $U(\Pi_p)$ is crucial:

- Direct and fast: $U_1 := U(\Pi_p)$.
- Strong: $U(\Pi_p)$ decomposes into nonadjacent cliques $U_j$.
- Mixed: Non-adjacent cliques + rest (see next slide).

# The $s - t$ Network, depending on $U(\Pi_p)$



# The $s - t$ Network, depending on $U(\Pi_p)$

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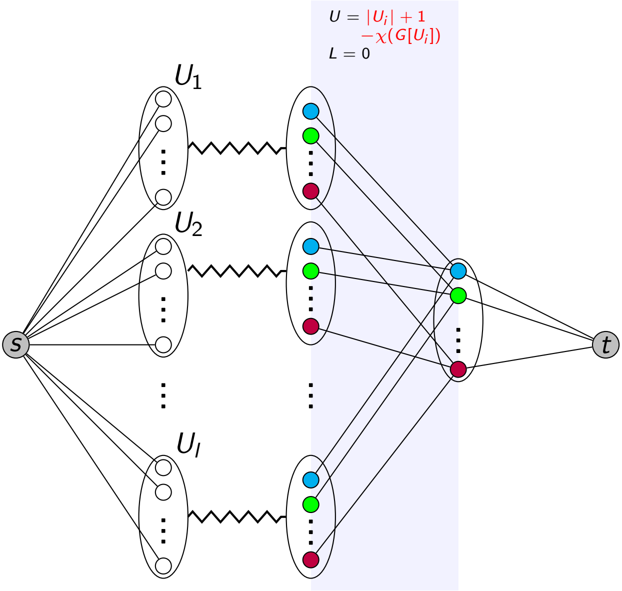
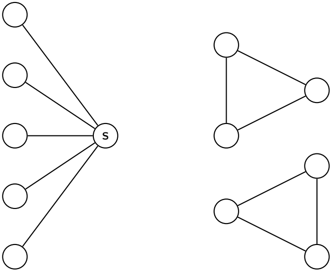


- It is $\chi_e(G) \leq 4$. How to see that $\chi_e(G) \neq 3$?




- Start DSATUR by coloring the center of the star.
  Continue with the satellites, the cliques are colored last.
- DSATUR without extended pruning rules: hundreds of nodes.
- With extended pruning rules: Solved in a single node.

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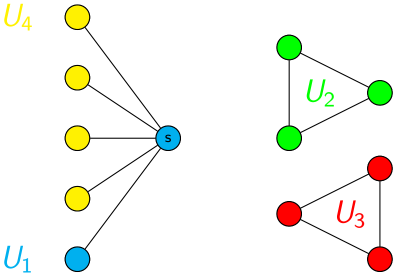


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# Computation Study: Setting

**Comparison:**

- DSATUR with extended pruning rules (EEQD) to DSATUR as presented in [4] (STD).
- The pruning rules are applied at every node.
- At first the pruning rule given by Th. 1 is evaluated. If no pruning occurs, the mixed approach is evaluated, i.e., the corresponding $s - t$ neworks are tested.
- EEQD will have at most as many B&B nodes as STD. We employ the decrease within these nodes as main criterion for improvement of EQDC over STD.
- Test on Erdős-Rényi $G(n, p)$ graphs (200) with $n \in \{40 + 5i \mid |i = 0, \ldots, 5\}$ and $p \in \{0.1 \cdot i \mid |i = 1, \ldots, 9\}$,
- Timelimit: 3600 seconds.

# Detailed Results I

Comparisons only on the instances both Alg. could solve.

| | p | Nodes | | Time | | | # Unsolved | |
|---|---|---|---|---|---|---|---|---|
| | | STD | EEQD | STD | EEQD | FF | STD | EEQD |
| n=40 | 0.1 | 107 | 57 | 0.00 | 0.00 | 0.00 | 0 | 0 |
| | 0.2 | 301 | 116 | 0.00 | 0.01 | 0.01 | 1 | 1 |
| | 0.3 | 734 | 248 | 0.00 | 0.03 | 0.02 | 0 | 0 |
| | 0.4 | 846000 | 721 | 1.17 | 0.06 | 0.04 | 2 | 0 |
| | 0.5 | 5277732 | 2269 | 9.96 | 0.17 | 0.13 | 0 | 0 |
| | 0.6 | 5670632 | 2278 | 10.97 | 0.17 | 0.13 | 2 | 0 |
| | 0.7 | 112556 | 601 | 0.15 | 0.09 | 0.06 | 0 | 0 |
| | 0.8 | 1005 | 393 | 0.00 | 0.05 | 0.03 | 2 | 0 |
| | 0.9 | 4598 | 39 | 0.01 | 0.01 | 0.00 | 2 | 2 |
| n=45 | 0.1 | 284 | 90 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.2 | 623 | 147 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.3 | 25479 | 1558 | 0.03 | 0.08 | 0.07 | 0 | 0 |
| | 0.4 | 7989770 | 557998 | 8.28 | 11.02 | 9.29 | 1 | 0 |
| | 0.5 | 12535143 | 6160 | 14.63 | 0.64 | 0.50 | 2 | 0 |
| | 0.6 | 9135279 | 3174 | 16.93 | 0.38 | 0.28 | 2 | 0 |
| | 0.7 | 44647230 | 2137 | 45.83 | 0.25 | 0.16 | 2 | 0 |
| | 0.8 | 4935 | 1006 | 0.01 | 0.15 | 0.09 | 0 | 0 |
| | 0.9 | 58836 | 83 | 0.08 | 0.02 | 0.01 | 1 | 1 |
| n=50 | 0.1 | 315 | 82 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.2 | 590799 | 679 | 1.06 | 0.08 | 0.06 | 0 | 0 |
| | 0.3 | 1558958 | 59274 | 2.37 | 2.34 | 1.84 | 0 | 0 |
| | 0.4 | 10014800 | 41453 | 16.52 | 2.03 | 1.71 | 1 | 0 |
| | 0.5 | 5727211 | 60236 | 8.97 | 5.51 | 4.23 | 3 | 0 |
| | 0.6 | 925567 | 12444 | 1.22 | 1.92 | 1.36 | 2 | 0 |
| | 0.7 | 38600548 | 10333 | 38.29 | 1.41 | 0.94 | 5 | 0 |
| | 0.8 | 39618 | 3111 | 0.06 | 0.67 | 0.41 | 0 | 0 |
| | 0.9 | 3966324 | 289 | 6.54 | 0.06 | 0.03 | 0 | 0 |

# Detailed Results I

Comparisons only on the instances both Alg. could solve.

| | p | Nodes | | Time | | | # Unsolved | |
|---|---|---|---|---|---|---|---|---|
| | | STD | EEQD | STD | EEQD | FF | STD | EEQD |
| n=40 | 0.1 | **107** | **57** | 0.00 | 0.00 | 0.00 | 0 | 0 |
| | 0.2 | **301** | **116** | 0.00 | 0.01 | 0.01 | 1 | 1 |
| | 0.3 | 734 | 248 | 0.00 | 0.03 | 0.02 | 0 | 0 |
| | 0.4 | 846000 | 721 | 1.17 | 0.06 | 0.04 | 2 | 0 |
| | 0.5 | 5277732 | 2269 | 9.96 | 0.17 | 0.13 | 0 | 0 |
| | 0.6 | 5670632 | 2278 | 10.97 | 0.17 | 0.13 | 2 | 0 |
| | 0.7 | 112556 | 601 | 0.15 | 0.09 | 0.06 | 0 | 0 |
| | 0.8 | 1005 | 393 | 0.00 | 0.05 | 0.03 | 2 | 0 |
| | 0.9 | 4598 | 39 | 0.01 | 0.01 | 0.00 | 2 | 2 |
| n=45 | 0.1 | 284 | 90 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.2 | 623 | 147 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.3 | 25479 | 1558 | 0.03 | 0.08 | 0.07 | 0 | 0 |
| | 0.4 | 7989770 | 557998 | 8.28 | 11.02 | 9.29 | 1 | 0 |
| | 0.5 | 12535143 | 6160 | 14.63 | 0.64 | 0.50 | 2 | 0 |
| | 0.6 | 9135279 | 3174 | 16.93 | 0.38 | 0.28 | 2 | 0 |
| | 0.7 | 44647230 | 2137 | 45.83 | 0.25 | 0.16 | 2 | 0 |
| | 0.8 | 4935 | 1006 | 0.01 | 0.15 | 0.09 | 0 | 0 |
| | 0.9 | 58836 | 83 | 0.08 | 0.02 | 0.01 | 1 | 1 |
| n=50 | 0.1 | 315 | 82 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.2 | 590799 | 679 | 1.06 | 0.08 | 0.06 | 0 | 0 |
| | 0.3 | 1558958 | 59274 | 2.37 | 2.34 | 1.84 | 0 | 0 |
| | 0.4 | 10014800 | 41453 | 16.52 | 2.03 | 1.71 | 1 | 0 |
| | 0.5 | 5727211 | 60236 | 8.97 | 5.51 | 4.23 | 3 | 0 |
| | 0.6 | 925567 | 12444 | 1.22 | 1.92 | 1.36 | 2 | 0 |
| | 0.7 | 38600548 | 10333 | 38.29 | 1.41 | 0.94 | 5 | 0 |
| | 0.8 | 39618 | 3111 | 0.06 | 0.67 | 0.41 | 0 | 0 |
| | 0.9 | 3966324 | 289 | 6.54 | 0.06 | 0.03 | 0 | 0 |

# Detailed Results I

Comparisons only on the instances both Alg. could solve.

| | p | Nodes | | Time | | | # Unsolved | |
|---|---|---|---|---|---|---|---|---|
| | | STD | EEQD | STD | EEQD | FF | STD | EEQD |
| n=40 | 0.1 | 107 | 57 | 0.00 | 0.00 | 0.00 | 0 | 0 |
| | 0.2 | 301 | 116 | 0.00 | 0.01 | 0.01 | 1 | 1 |
| | 0.3 | 734 | 248 | 0.00 | 0.03 | 0.02 | 0 | 0 |
| | 0.4 | **846000** | **721** | 1.17 | 0.06 | 0.04 | 2 | 0 |
| | 0.5 | **5277732** | **2269** | 9.96 | 0.17 | 0.13 | 0 | 0 |
| | 0.6 | **5670632** | **2278** | 10.97 | 0.17 | 0.13 | 2 | 0 |
| | 0.7 | **112556** | **601** | 0.15 | 0.09 | 0.06 | 0 | 0 |
| | 0.8 | 1005 | 393 | 0.00 | 0.05 | 0.03 | 2 | 0 |
| | 0.9 | 4598 | 39 | 0.01 | 0.01 | 0.00 | 2 | 2 |
| n=45 | 0.1 | 284 | 90 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.2 | 623 | 147 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.3 | 25479 | 1558 | 0.03 | 0.08 | 0.07 | 0 | 0 |
| | 0.4 | 7989770 | 557998 | 8.28 | 11.02 | 9.29 | 1 | 0 |
| | 0.5 | 12535143 | 6160 | 14.63 | 0.64 | 0.50 | 2 | 0 |
| | 0.6 | 9135279 | 3174 | 16.93 | 0.38 | 0.28 | 2 | 0 |
| | 0.7 | 44647230 | 2137 | 45.83 | 0.25 | 0.16 | 2 | 0 |
| | 0.8 | 4935 | 1006 | 0.01 | 0.15 | 0.09 | 0 | 0 |
| | 0.9 | 58836 | 83 | 0.08 | 0.02 | 0.01 | 1 | 1 |
| n=50 | 0.1 | 315 | 82 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.2 | 590799 | 679 | 1.06 | 0.08 | 0.06 | 0 | 0 |
| | 0.3 | 1558958 | 59274 | 2.37 | 2.34 | 1.84 | 0 | 0 |
| | 0.4 | 10014800 | 41453 | 16.52 | 2.03 | 1.71 | 1 | 0 |
| | 0.5 | 5727211 | 60236 | 8.97 | 5.51 | 4.23 | 3 | 0 |
| | 0.6 | 925567 | 12444 | 1.22 | 1.92 | 1.36 | 2 | 0 |
| | 0.7 | 38600548 | 10333 | 38.29 | 1.41 | 0.94 | 5 | 0 |
| | 0.8 | 39618 | 3111 | 0.06 | 0.67 | 0.41 | 0 | 0 |
| | 0.9 | 3966324 | 289 | 6.54 | 0.06 | 0.03 | 0 | 0 |

# Detailed Results I

Comparisons only on the instances both Alg. could solve.

| | p | Nodes | | Time | | | # Unsolved | |
|---|---|---|---|---|---|---|---|---|
| | | STD | EEQD | STD | EEQD | FF | STD | EEQD |
| n=40 | 0.1 | 107 | 57 | 0.00 | 0.00 | 0.00 | 0 | 0 |
| | 0.2 | 301 | 116 | 0.00 | 0.01 | 0.01 | 1 | 1 |
| | 0.3 | 734 | 248 | 0.00 | 0.03 | 0.02 | 0 | 0 |
| | 0.4 | 846000 | 721 | 1.17 | 0.06 | 0.04 | 2 | 0 |
| | 0.5 | 5277732 | 2269 | 9.96 | 0.17 | 0.13 | 0 | 0 |
| | 0.6 | 5670632 | 2278 | 10.97 | 0.17 | 0.13 | 2 | 0 |
| | 0.7 | 112556 | 601 | 0.15 | 0.09 | 0.06 | 0 | 0 |
| | 0.8 | **1005** | **393** | 0.00 | 0.05 | 0.03 | 2 | 0 |
| | 0.9 | **4598** | **39** | 0.01 | 0.01 | 0.00 | 2 | 2 |
| n=45 | 0.1 | 284 | 90 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.2 | 623 | 147 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.3 | 25479 | 1558 | 0.03 | 0.08 | 0.07 | 0 | 0 |
| | 0.4 | 7989770 | 557998 | 8.28 | 11.02 | 9.29 | 1 | 0 |
| | 0.5 | 12535143 | 6160 | 14.63 | 0.64 | 0.50 | 2 | 0 |
| | 0.6 | 9135279 | 3174 | 16.93 | 0.38 | 0.28 | 2 | 0 |
| | 0.7 | 44647230 | 2137 | 45.83 | 0.25 | 0.16 | 2 | 0 |
| | 0.8 | 4935 | 1006 | 0.01 | 0.15 | 0.09 | 0 | 0 |
| | 0.9 | 58836 | 83 | 0.08 | 0.02 | 0.01 | 1 | 1 |
| n=50 | 0.1 | 315 | 82 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.2 | 590799 | 679 | 1.06 | 0.08 | 0.06 | 0 | 0 |
| | 0.3 | 1558958 | 59274 | 2.37 | 2.34 | 1.84 | 0 | 0 |
| | 0.4 | 10014800 | 41453 | 16.52 | 2.03 | 1.71 | 1 | 0 |
| | 0.5 | 5727211 | 60236 | 8.97 | 5.51 | 4.23 | 3 | 0 |
| | 0.6 | 925567 | 12444 | 1.22 | 1.92 | 1.36 | 2 | 0 |
| | 0.7 | 38600548 | 10333 | 38.29 | 1.41 | 0.94 | 5 | 0 |
| | 0.8 | 39618 | 3111 | 0.06 | 0.67 | 0.41 | 0 | 0 |
| | 0.9 | 3966324 | 289 | 6.54 | 0.06 | 0.03 | 0 | 0 |

# Detailed Results I

Comparisons only on the instances both Alg. could solve.

| | p | Nodes | | Time | | | # Unsolved | |
|---|---|---|---|---|---|---|---|---|
| | | STD | EEQD | STD | EEQD | FF | STD | EEQD |
| n=40 | 0.1 | 107 | 57 | 0.00 | 0.00 | 0.00 | 0 | 0 |
| | 0.2 | 301 | 116 | 0.00 | 0.01 | 0.01 | 1 | 1 |
| | 0.3 | 734 | 248 | 0.00 | 0.03 | 0.02 | 0 | 0 |
| | 0.4 | 846000 | 721 | 1.17 | 0.06 | 0.04 | 2 | 0 |
| | 0.5 | 5277732 | 2269 | 9.96 | 0.17 | 0.13 | 0 | 0 |
| | 0.6 | 5670632 | 2278 | 10.97 | 0.17 | 0.13 | 2 | 0 |
| | 0.7 | 112556 | 601 | 0.15 | 0.09 | 0.06 | 0 | 0 |
| | 0.8 | 1005 | 393 | 0.00 | 0.05 | 0.03 | 2 | 0 |
| | 0.9 | 4598 | 39 | 0.01 | 0.01 | 0.00 | 2 | 2 |
| n=45 | 0.1 | 284 | 90 | 0.00 | 0.01 | 0.01 | **0** | **0** |
| | 0.2 | 623 | 147 | 0.00 | 0.01 | 0.01 | **0** | **0** |
| | 0.3 | 25479 | 1558 | 0.03 | 0.08 | 0.07 | **0** | **0** |
| | 0.4 | 7989770 | 557998 | 8.28 | 11.02 | 9.29 | **1** | **0** |
| | 0.5 | 12535143 | 6160 | 14.63 | 0.64 | 0.50 | **2** | **0** |
| | 0.6 | 9135279 | 3174 | 16.93 | 0.38 | 0.28 | **2** | **0** |
| | 0.7 | 44647230 | 2137 | 45.83 | 0.25 | 0.16 | **2** | **0** |
| | 0.8 | 4935 | 1006 | 0.01 | 0.15 | 0.09 | **0** | **0** |
| | 0.9 | 58836 | 83 | 0.08 | 0.02 | 0.01 | **1** | **1** |
| n=50 | 0.1 | 315 | 82 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.2 | 590799 | 679 | 1.06 | 0.08 | 0.06 | 0 | 0 |
| | 0.3 | 1558958 | 59274 | 2.37 | 2.34 | 1.84 | 0 | 0 |
| | 0.4 | 10014800 | 41453 | 16.52 | 2.03 | 1.71 | 1 | 0 |
| | 0.5 | 5727211 | 60236 | 8.97 | 5.51 | 4.23 | 3 | 0 |
| | 0.6 | 925567 | 12444 | 1.22 | 1.92 | 1.36 | 2 | 0 |
| | 0.7 | 38600548 | 10333 | 38.29 | 1.41 | 0.94 | 5 | 0 |
| | 0.8 | 39618 | 3111 | 0.06 | 0.67 | 0.41 | 0 | 0 |
| | 0.9 | 3966324 | 289 | 6.54 | 0.06 | 0.03 | 0 | 0 |

# Detailed Results I

Comparisons only on the instances **both** Alg. could solve.

| | p | Nodes | | Time | | | # Unsolved | |
|---|---|---|---|---|---|---|---|---|
| | | STD | EEQD | STD | EEQD | FF | STD | EEQD |
| n=40 | 0.1 | 107 | 57 | 0.00 | 0.00 | 0.00 | 0 | 0 |
| | 0.2 | 301 | 116 | 0.00 | 0.01 | 0.01 | 1 | 1 |
| | 0.3 | 734 | 248 | 0.00 | 0.03 | 0.02 | 0 | 0 |
| | 0.4 | 846000 | 721 | 1.17 | 0.06 | 0.04 | 2 | 0 |
| | 0.5 | 5277732 | 2269 | 9.96 | 0.17 | 0.13 | 0 | 0 |
| | 0.6 | 5670632 | 2278 | 10.97 | 0.17 | 0.13 | 2 | 0 |
| | 0.7 | 112556 | 601 | 0.15 | 0.09 | 0.06 | 0 | 0 |
| | 0.8 | 1005 | 393 | 0.00 | 0.05 | 0.03 | 2 | 0 |
| | 0.9 | 4598 | 39 | 0.01 | 0.01 | 0.00 | 2 | 2 |
| n=45 | 0.1 | 284 | 90 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.2 | 623 | 147 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.3 | 25479 | 1558 | 0.03 | 0.08 | 0.07 | 0 | 0 |
| | 0.4 | 7989770 | 557998 | 8.28 | 11.02 | 9.29 | 1 | 0 |
| | 0.5 | 12535143 | 6160 | 14.63 | 0.64 | 0.50 | 2 | 0 |
| | 0.6 | 9135279 | 3174 | 16.93 | 0.38 | 0.28 | 2 | 0 |
| | 0.7 | 44647230 | 2137 | 45.83 | 0.25 | 0.16 | 2 | 0 |
| | 0.8 | 4935 | 1006 | 0.01 | 0.15 | 0.09 | 0 | 0 |
| | 0.9 | 58836 | 83 | 0.08 | 0.02 | 0.01 | 1 | 1 |
| n=50 | 0.1 | 315 | 82 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.2 | 590799 | 679 | 1.06 | 0.08 | 0.06 | 0 | 0 |
| | 0.3 | 1558958 | 59274 | 2.37 | 2.34 | 1.84 | 0 | 0 |
| | 0.4 | 10014800 | 41453 | **16.52** | **2.03** | 1.71 | 1 | 0 |
| | 0.5 | 5727211 | 60236 | 8.97 | 5.51 | 4.23 | 3 | 0 |
| | 0.6 | 925567 | 12444 | 1.22 | 1.92 | 1.36 | 2 | 0 |
| | 0.7 | 38600548 | 10333 | 38.29 | 1.41 | 0.94 | 5 | 0 |
| | 0.8 | 39618 | 3111 | **0.06** | **0.67** | 0.41 | 0 | 0 |
| | 0.9 | 3966324 | 289 | 6.54 | 0.06 | 0.03 | 0 | 0 |

# Detailed Results II

Comparisons only on the instances both Alg. could solve.

| | p | Nodes STD | Nodes EEQD | Time STD | Time EEQD | Time FF | # Unsolved STD | # Unsolved EEQD |
|---|---|---|---|---|---|---|---|---|
| n=55 | 0.1 | 314 | 95 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.2 | 167703 | 13966 | 0.33 | 0.62 | 0.52 | 0 | 0 |
| | 0.3 | 5299 | 3964 | 0.01 | 0.97 | 0.80 | 0 | 0 |
| | 0.4 | 4892479 | 148580 | 7.37 | 19.57 | 15.57 | 0 | 0 |
| | 0.5 | 15846172 | 109874 | 23.13 | 15.23 | 11.71 | 2 | 2 |
| | 0.6 | 9873776 | 147986 | 10.36 | 17.00 | 12.09 | 2 | 0 |
| | 0.7 | 47561545 | 54194 | 60.68 | 9.12 | 6.17 | 17 | 0 |
| | 0.8 | 194259 | 8663 | 0.28 | 2.06 | 1.30 | 0 | 0 |
| | 0.9 | 53112985 | 875 | 70.83 | 0.17 | 0.08 | 3 | 0 |
| n=60 | 0.1 | 931 | 156 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.2 | 161554 | 3882 | 0.33 | 1.07 | 0.91 | 0 | 0 |
| | 0.3 | 9711671 | 161002 | 14.09 | 18.03 | 14.00 | 1 | 0 |
| | 0.4 | 7976291 | 95606 | 14.57 | 17.13 | 13.75 | 6 | 5 |
| | 0.5 | 15669370 | 241625 | 28.69 | 53.77 | 42.33 | 5 | 4 |
| | 0.6 | 129671603 | 208994 | 136.79 | 47.23 | 34.48 | 19 | 1 |
| | 0.7 | 7936038 | 214101 | 12.14 | 47.74 | 32.70 | 18 | 0 |
| | 0.8 | 9118364 | 25454 | 13.95 | 7.25 | 4.57 | 0 | 0 |
| | 0.9 | 39384788 | 5291 | 53.11 | 1.24 | 0.58 | 23 | 0 |
| n=65 | 0.1 | 1606 | 206 | 0.00 | 0.02 | 0.01 | 0 | 0 |
| | 0.2 | 159866 | 3680 | 0.22 | 1.02 | 0.88 | 0 | 0 |
| | 0.3 | 648497 | 114286 | 1.04 | 25.89 | 21.86 | 0 | 0 |
| | 0.4 | 8994787 | 972954 | 20.72 | 141.47 | 116.36 | 1 | 1 |
| | 0.5 | 47689077 | 2506469 | 52.71 | 517.87 | 397.83 | 13 | 4 |
| | 0.6 | 64290978 | 2114960 | 80.67 | 476.17 | 365.01 | 16 | 0 |
| | 0.7 | 11847345 | 881496 | 14.71 | 205.96 | 145.65 | 18 | 0 |
| | 0.8 | 104398385 | 282265 | 110.00 | 55.65 | 34.91 | 2 | 0 |
| | 0.9 | 14647901 | 26359 | 17.35 | 6.15 | 2.92 | 19 | 0 |
| **Av. (Sum)** | | **13,920,351** | **168,777** | **17.17** | **31.77** | **17.22** | **4** | **0** |

# Detailed Results II

Comparisons only on the instances both Alg. could solve.

| | p | Nodes | | Time | | | # Unsolved | |
|---|---|---|---|---|---|---|---|---|
| | | STD | EEQD | STD | EEQD | FF | STD | EEQD |
| n=55 | 0.1 | 314 | 95 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.2 | **167703** | **13966** | 0.33 | 0.62 | 0.52 | 0 | 0 |
| | 0.3 | 5299 | 3964 | 0.01 | 0.97 | 0.80 | 0 | 0 |
| | 0.4 | 4892479 | 148580 | 7.37 | 19.57 | 15.57 | 0 | 0 |
| | 0.5 | 15846172 | 109874 | 23.13 | 15.23 | 11.71 | 2 | 2 |
| | 0.6 | 9873776 | 147986 | 10.36 | 17.00 | 12.09 | 2 | 0 |
| | 0.7 | 47561545 | 54194 | 60.68 | 9.12 | 6.17 | 17 | 0 |
| | 0.8 | 194259 | 8663 | 0.28 | 2.06 | 1.30 | 0 | 0 |
| | 0.9 | 53112985 | 875 | 70.83 | 0.17 | 0.08 | 3 | 0 |
| n=60 | 0.1 | 931 | 156 | 0.00 | 0.01 | 0.01 | 0 | 0 |
| | 0.2 | 161554 | 3882 | 0.33 | 1.07 | 0.91 | 0 | 0 |
| | 0.3 | 9711671 | 161002 | 14.09 | 18.03 | 14.00 | 1 | 0 |
| | 0.4 | 7976291 | 95606 | 14.57 | 17.13 | 13.75 | 6 | 5 |
| | 0.5 | 15669370 | 241625 | **28.69** | **53.77** | 42.33 | 5 | 4 |
| | 0.6 | 129671603 | 208994 | **136.79** | **47.23** | 34.48 | 19 | 1 |
| | 0.7 | 7936038 | 214101 | 12.14 | 47.74 | 32.70 | 18 | 0 |
| | 0.8 | 9118364 | 25454 | 13.95 | 7.25 | 4.57 | 0 | 0 |
| | 0.9 | 39384788 | 5291 | 53.11 | 1.24 | 0.58 | 23 | 0 |
| n=65 | 0.1 | 1606 | 206 | 0.00 | 0.02 | 0.01 | 0 | 0 |
| | 0.2 | 159866 | 3680 | 0.22 | 1.02 | 0.88 | 0 | 0 |
| | 0.3 | 648497 | 114286 | 1.04 | 25.89 | 21.86 | 0 | 0 |
| | 0.4 | 8994787 | 972954 | 20.72 | 141.47 | 116.36 | 1 | 1 |
| | 0.5 | 47689077 | 2506469 | 52.71 | 517.87 | 397.83 | 13 | 4 |
| | 0.6 | 64290978 | 2114960 | 80.67 | 476.17 | 365.01 | 16 | 0 |
| | 0.7 | 11847345 | 881496 | 14.71 | 205.96 | 145.65 | 18 | 0 |
| | 0.8 | 104398385 | 282265 | 110.00 | 55.65 | 34.91 | 2 | 0 |
| | 0.9 | 14647901 | 26359 | 17.35 | 6.15 | 2.92 | 19 | 0 |
| **Av. (Sum)** | | **13,920,351** | **168,777** | **17.17** | **31.77** | **17.22** | **4** | **0** |

# Comments

- The pruning has a dramatic effect on the size of the B&B tree.
  EEQD needs on average 1.2% of the nodes of STD.
  This factor can be as small as 0.0016% ($n = 55$, $p = 0,9$).

- EEQD takes about 1.85 the time of STD, as
  in each B&B node 2 flow problems are solved
  (up to two times 2.5 million for $n = 65$, $p = 0.5$).

- Runtime improvement for 23 out of 54 classes of the instances.

- STD has much more instances which hit the time limit compared to EEQD (193 vs. 21)

- **Conclusion:** EEQD is slower but much more stable than STD.

# Performance Over Time

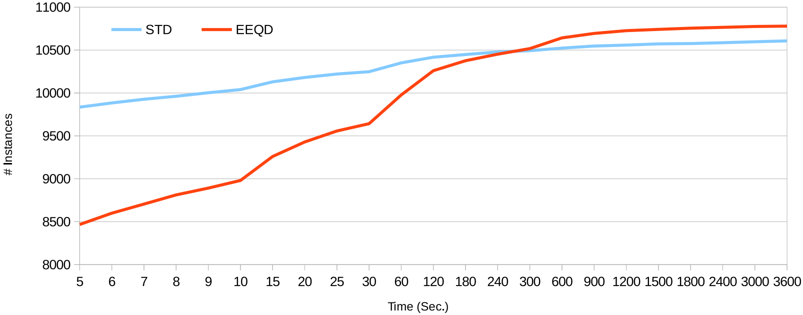



Figure : Instances solved within a fixed time frame by STD and EEQD.

**Trend:** Between 5 and 240 seconds, STD solves more instances. Above 240 seconds, the opposite holds.
Either an instance can be sovled in e.g., 600 seconds, or not at all.

# Conclusion

We considered a DSATUR algorithm for equitable coloring.

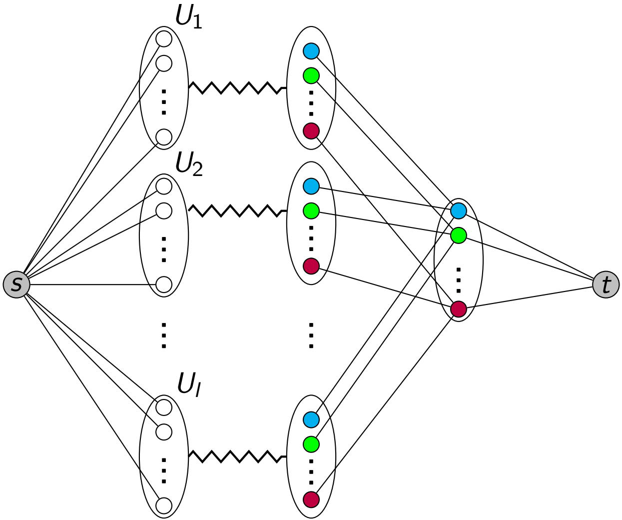
**Results:**

- New Pruning rules, based on network flows.
- Very effective w.r.t. the number of B&B nodes.
- Mixed effect on solution time:
  On average slower, but solves more instances.

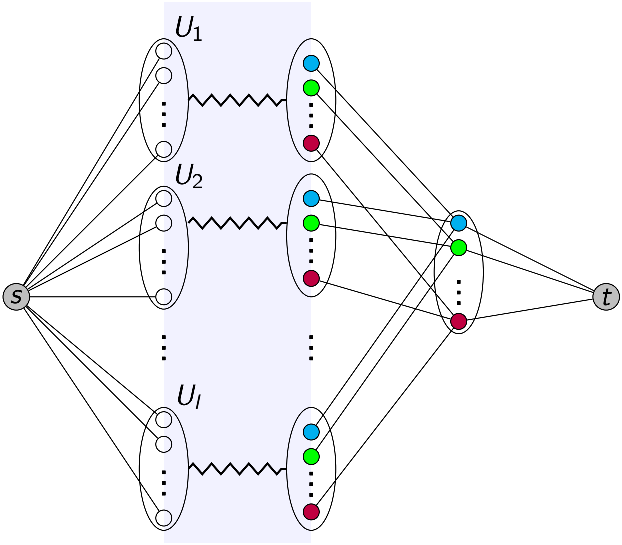
**Future Work:**

- More efficient implementation.
- Evaluate different applications of the pruning rules.
- Derive conditions which can be evaluated via formulae.

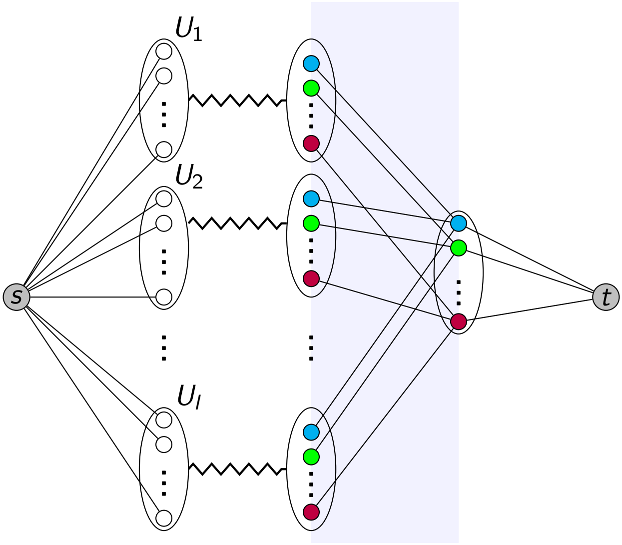
# The $s - t$ Network, extract formulae



# The $s - t$ Network, extract formulae



# The $s-t$ Network, extract formulae



# A Flow Based Pruning Scheme For Enumerative Equitable Coloring Algorithms

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