Conic programming bounds for structured combinatorial problems

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Cones at a glance

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Primal cone ($\mathcal{K}$) - dual cone ($\mathcal{K}^*$)

$$\mathcal{K}^* := \{s \in \mathbb{R}^n : \langle s, x \rangle := s^T x \geq 0 \text{ for all } x \in \mathcal{K}\}$$

$$\mathcal{K}^* := \{S \in \mathbb{R}^{n \times n} : \langle S, X \rangle := \mathrm{trace}(SX) \geq 0 \text{ for all } X \in \mathcal{K}\}$$

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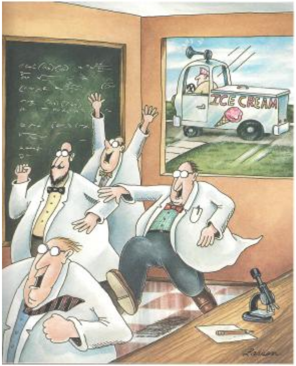
"Famous" dual cones

$$(\mathbb{R}^n_+)^* = \mathbb{R}^n_+ \quad (\mathcal{S}_n^+)^* = \mathcal{S}_n^+$$

$$\mathcal{C}^* := \mathrm{cone}\{xx^T : x \geq 0\}$$

Linearization over suitable cones



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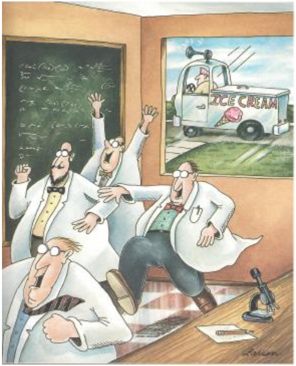

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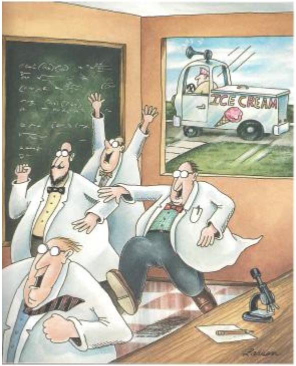

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$$X = xx^T \quad \Leftrightarrow \quad \mathrm{rank}(X) = 1 \text{ and } X \succeq 0.$$

Standard Quadratic Optimization Problem

$$\begin{aligned} \min \quad & x^T A x + 2c^T x \\ \text{s.t.} \quad & e^T x = 1 \\ & x \geq 0. \end{aligned}$$

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... because xx^T with $x \geq 0$ are the extreme rays of $\mathcal{C}^*$...

$$\begin{aligned} \min \quad & \langle Q, X \rangle \\ \text{s.t.} \quad & \langle J, X \rangle = 1 \\ & X \in \mathcal{C}^*. \end{aligned}$$

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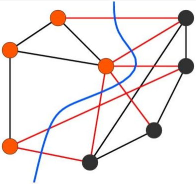
First paper to describe relaxations of NP-hard combinatorial optimization problems

A.J. Quist, E. de Klerk, C. Roos and T. Terlaky. Copositive relaxation for general quadratic programming. *Optimization Methods and Software*, **9**:185–208, 1998.

Max-cut problem



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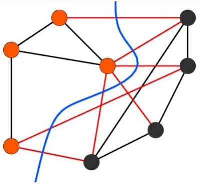

$V = [S_1 S_2]$

$$\mathrm{MC} := \max \quad \sum_{i,j=1}^{n} W_{ij} \left(\frac{1 - x_i x_j}{4} \right)$$
$$\text{s.t.} \quad x_i \in \{-1, 1\}, \quad i \in V.$$

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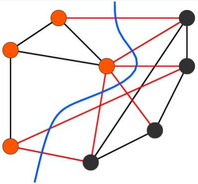

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$x \in \{-1,1\}^n$ and $X := xx^T$

$$\begin{aligned} \mathrm{MC} = \max \quad & \frac{1}{4}\mathrm{trace}(W(J - X)) \\ \text{s.t.} \quad & \mathrm{diag}(\mathrm{X}) = \mathrm{e}, \\ & X \succeq 0, \\ & \mathrm{rank}(X) = 1. \end{aligned}$$

Conic programming

- For given symmetric $n \times n$ matrices $A_0, \ldots, A_m$ and $b \in \mathbb{R}^m$, the standard conic programing problem is defined as:

$$\begin{aligned} \min \quad & \langle A_0, X \rangle \\ \text{s.t.} \quad & \langle A_i, X \rangle = b_i, \quad i = 1, ..., m \\ & X \in \mathcal{K}. \end{aligned}$$

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- if $\mathcal{K} = \mathcal{S}_n^+$ we talk about semidefinite programming
- if $\mathcal{K} = \mathcal{C}$ we talk about copositive programming

Copositivity and combinatorial problems

Reformulation of the standard quadratic optimization problem

I.M. Bomze, M. Dür, E. de Klerk, C. Roos, A.J. Quist, and T. Terlaky. On copositive programming and standard quadratic optimization problems. *Journal of Global Optimization*, **18**: 301–320, 2000.

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Reformulation of the binary and continuous nonconvex quadratic programs

S. Burer. On the copositive representation of binary and continuous nonconvex quadratic programs. *Mathematical Programming A*, **120(2)**:479–495, 2009.

Optimizing over $\mathcal{C}$

- all the combinatorial problems mentioned before are "hard"
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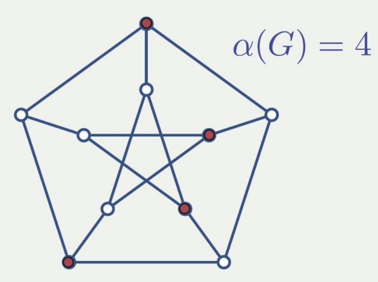

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Good news :)

Exploit the symmetry coming from the structure of the problem.

Approximation hierarchies (ctd...)

Parrilo

P. Parrilo. *Structured Semidefinite programming and Algebraic Geometry Methods in Robustness and Optimization. PhD thesis, California Institute of Technology, Pasadena, CA, USA, 2000.*

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Pena - Vera - Zuluaga

J. Pena, J. Vera, and L.F. Zuluaga. Computing the stability number of a graph via linear and semidefinite programming. *SIAM Journal on Optimization*, **18(1)**:87–105, 2007.

Pena - Vera - Zuluaga hierarchy

Given $\beta \in \mathbb{N}^n$, $|\beta| := \beta_1 + ... + \beta_n$, $x^\beta := x^{\beta_1} \cdots x^{\beta_n}$

$$\mathcal{E}^r := \left\{ \sum_{\beta \in \mathbb{N}^n, |\beta| = r} x^\beta x^T (P_\beta + N_\beta) x \ : \ P_\beta \in \mathcal{S}_n^+, N_\beta \in \mathcal{N} \right\}$$

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$\mathcal{K}^0$ - dual of doubly nonnegative matrices / $\mathcal{K}^1$

$$\mathcal{K}^0 := \left\{ M \in \mathcal{S} \;:\; x^T M x = x^T (P_\beta + N_\beta) x \right\},$$

$$\mathcal{K}^1 := \left\{ M \in \mathcal{S} \;:\; x^T M x (\sum_{i=1}^n x_i) = \sum_{i=1}^n x_i x^T (P_i + N_i) x \right\}.$$

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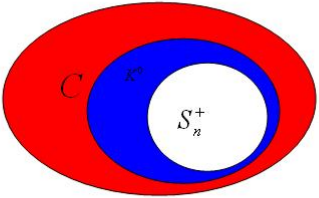
- $\mathcal{S}_n^+ + \mathcal{N} = \mathcal{K}^0 \subset \mathcal{K}^1 \subset \ldots \subset \mathcal{C}$



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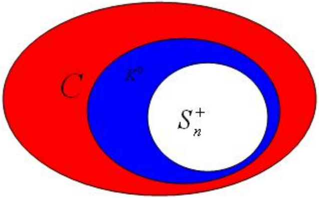

Relaxation

$$\alpha(G) \leq \min \{\lambda \ : \ \lambda(A+I) - J \in \mathcal{K}^r\}.$$

$\lambda(A + I) - J \in \mathcal{K}^1$

- denote $\lambda(A + I) - J := M$

Recall...

$$\mathcal{K}^1 := \left\{ M \in \mathcal{S} \ : \ x^T M x (\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} x_i x^T (P_i + N_i) x \right\}.$$

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Equivalent to:

$$\mathcal{K}^1 := \left\{ M \in \mathcal{S} \ : \ x^T M x (\sum_{i=1}^{n} x_i) \geq \sum_{i=1}^{n} x_i x^T P_i x \right\}.$$

$\lambda(A+I)-J\in\mathcal{K}^1$

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Recall...

$$\mathcal{K}^1 := \left\{ M\in\mathcal{S} \;:\; x^TMx(\sum_{i=1}^n x_i) = \sum_{i=1}^n x_i x^T(P_i+N_i)x \right\}.$$

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Thus:

$$\alpha(G) \leq \min\left\{ \lambda \;:\; x^TMx(\sum_{i=1}^n x_i) \geq \sum_{i=1}^n x_i x^T P_i x,\; P_i\succeq 0 \right\}.$$

ALGEBRAIC DETOUR !!!

Matrix algebras

Definition

A set $\mathcal{A} \subseteq \mathbb{C}^{n\times n}$ (resp. $\mathbb{R}^{n\times n}$) is called a matrix $*$-algebra over $\mathbb{C}$ (resp. $\mathbb{R}$) if, for all $X, Y \in \mathcal{A}$:

- $\alpha X + \beta Y \in \mathcal{A}\ \forall \alpha, \beta \in \mathbb{C}$ (resp. $\mathbb{R}$);
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Denote the basis of $\mathcal{A}$ by $\{B_1, B_2, ..., B_d\}$.

Example

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Form of a circulant matrix R

$$R = \begin{pmatrix} r_0 & r_1 & r_2 & . & . & r_{n-1} \\ r_{n-1} & r_0 & r_1 & r_2 & . & . \\ . & . & r_0 & r_1 & r_2 & . \\ . & . & . & . & . & . \\ r_2 & . & . & . & r_0 & r_1 \\ r_1 & r_2 & . & . & . & r_0 \end{pmatrix}$$

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- Each row is a cyclic shift of the row above it, i.e:

$$R_{ij} := r_{(j-i) \ mod \ n}.$$

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Further reading:

R.M. Gray. Toeplitz and Circulant Matrices: A review. *Foundation and Trends in Comunications and Information Theory*, **2**(3):155-239, 2006. Available online.

Example (ctd.)

- A basis for the symmetric 4×4 circulant matrices is

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Algebraic Symmetry for SDP Problems

Assumption

$M := \lambda(A + I) - J$ belongs to some matrix $\mathbb{C}*$-algebra $\mathcal{A}$ of dimension d, having the basis: $\{B_1, B_2, ..., B_d\}$.

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$M := \lambda(A + I) - J$ belongs to some matrix $\mathbb{C}*$-algebra $\mathcal{A}$ of dimension d, having the basis: $\{B_1, B_2, ..., B_d\}$.

Theorem

If the SDP problem has a optimal primal-dual solution then there exists an optimal primal-dual solution (X^*, y^*, S^*) such that $X^* \in \mathcal{A}$, $y \in \mathbb{R}^m$ and $S^* \in \mathcal{A}$.

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- $M = \sum_{i=1}^{d} m_i B_i$, for some $m_i \in \mathbb{C}$ (resp. $\mathbb{R}$).
- One obtain significant reductions when $\mathcal{A}$ is low dimensional.
- For the 4×4 circulant matrices we have: $n = 4$ and $d = 3$.

Canonical block diagonalization of a matrix *-algebra $\mathcal{A}$

Theorem (Wedderburn)

Assume $\mathcal{A}$ is a matrix *-algebra over $\mathbb{C}$ that contains I. Then there is a unitary Q ($Q^*Q = I$) and some integer s such that

$$Q^*\mathcal{A}Q = \begin{pmatrix} \mathcal{A}_1 & 0 & \dots & 0 \\ 0 & \mathcal{A}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{A}_s \end{pmatrix},$$

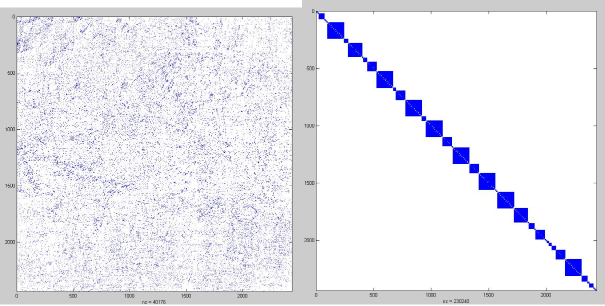
where each $\mathcal{A}_i$ takes the form

$$\mathcal{A}_i = \left\{ \begin{pmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A \end{pmatrix} \mid A \in \mathbb{C}^{n_i \times n_i} \right\},$$

for some integers n_i, $i = 1, ..., s$.

Smaller data matrices

- smaller data matrices yield numerical tractability of the original problem



The algebraic detour is over!

K. Murota, Y. Kanno, M. Kojima and S. Kojima: A numerical algorithm for block-diagonal decomposition of matrix *-algebras with application to semidefinite programming. *Japan Journal of Industrial and Applied Mathematics*, **27(1)**:125–160, 2010.

E. de Klerk, C. Dobre, D.V. Pasechnik: Numerical block diagonalization of matrix $*$-algebras with application to semidefinite programming. *Mathematical Programming B*, **129**:91–111, 2011.

Symmetry reduction

Recall...

$$\alpha(G) \leq \min \left\{ \lambda \ : \ x^T M x (\sum_{i=1}^{n} x_i) \geq \sum_{i=1}^{n} x_i x^T P_i x, \ P_i \succeq 0 \right\}.$$

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If aut(A) is transitive then the problem is equivalent to:

$$\alpha(G) \leq \min\left\{\lambda \ : \ x^T M x(\sum_{i=1}^n x_i) \geq x_1 x^T P x, \ P \succeq 0\right\},$$

where $P \in \mathrm{cent\ stab}_{\mathrm{aut}(A)}(1)$.

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Symmetry reduction (ctd...)

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$M = \sum_{i=1}^{d} m_i B_i$ and $P = \sum_{i=1}^{d_1} p_i D_i$

$$\alpha(G) \leq \min\left\{\lambda \ : \ \mathrm{Lone} * m \geq \mathrm{Lmany} * p, \ \sum_{i=1}^{d_1} p_i \oplus_{k=1}^{s} D_i^k \succeq 0\right\}.$$

Toy-example: 4-cycle (square)

Adjacency matrix - A

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

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Basis cent aut(A)

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \; B_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

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$$D_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, D_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, D_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

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$$Lone = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix},\ Lmany = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

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$n = 4,\ d = 3;\ d_1 = 7$

$$\alpha(C_4) \le \min\left\{\lambda\ :\ \mathrm{Lone} * m \ge \mathrm{Lmany} * p,\ \sum_{i=1}^{d_1} p_i D_i \succeq 0\right\}.$$

Example - The Crossing Number

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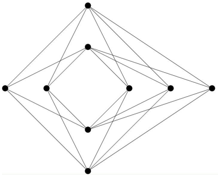

Figure : Drawing of $\mathcal{K}_{4,5}$ with 8 crossings

Definition

The crossing number $cr(G)$ of a graph G is the minimum number of pairwise intersection of edges in a drawing of G in the plane.

The Crossing Number of Complete Bipartite Graphs

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$$cr(K_{m,n}) \leq Z(m,n)$$

de Klerk et al. - 2008

$$cr(K_{m,n}) \geq \frac{n}{2} \left(n \max\{t \ : \ Q_{(m-1)!} - tJ_{(m-1)!} \in \mathcal{C}\} - \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{m}{2} \rfloor \right).$$

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Computing the Symmetry Reduction over $\mathcal{K}^1$

$n = 720,\ d = 78;\ d_1 = 19305$

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- The routines are implemented in Matlab.
- The problem is solved in less than four hours using SDPT3 in Coral lab at Lehigh University - 32 GB of RAM and 16 AMD Opteron 2.0 GHz Processor.

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- Adapt the techniques for other combinatorial problems.

Thank you for your attention !