Combinatorial Optimization inspired by Uncertainties

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Take away message

Uncertainties complicates Optimization

but

understanding the complexity increase helps (and is fun)

- **Case I:** developing *polyhedral theory* further
- **Case II:** reformulating to known problems
- **Case III:** determining *complexity border*

Joint works with Christina Büsing, Timo Gersing, Alexandra Grub, Manuel Kutschka, Wlademar Laube, Nils Spiekermann, Martin Tieves
1. Case I: Combinatorial Optimization under Uncertainty
2. Case II: Uncertainty-driven Generalizations
4. Concluding Remarks
Motivation: Bandwidth Packing Problem

Given network topology, link dimensioning, demands

Find routing

Observations:
- single path routing
- binary decision on single link → 0-1 Knapsack Problem
- demand values are uncertain
Motivation: Bandwidth Packing Problem

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Robust Optimization according to Ben-Tal and Nemirovski:

**Uncertain Linear Program**

An Uncertain Linear Optimization problem (ULO) is a collection of linear optimization problems (instances)

\[
\left\{ \min \{ c^T x : Ax \leq b \} \right\}_{(c,A,b) \in U}
\]

where all input data stems from an uncertainty set \( U \subset \mathbb{R}^n \times \mathbb{R}^{m \times n} \times \mathbb{R}^m \).

**Robust Knapsack Problem**

\[
\max \left\{ c^T x : \{ a^T x \leq b, x \in \{0, 1\}^n \} \right\}_{a \in U}
\]

How to define \( U \)?
How to define the uncertainty set?

- Uncertainty set is an *ellipsoid*, e.g.,

\[ \mathcal{U} = \{ a \in \mathbb{R}^n : \| a - \bar{a} \| < \kappa \} \]

- Uncertainty set is a *polyhedron*, e.g.,

\[ \mathcal{U} = \{ a \in \mathbb{R}^n : D \cdot a \leq d \} \]

with \( D \in \mathbb{R}^{k \times n}, \ d \in \mathbb{R}^k \) for some \( k \in \mathbb{N} \).

**equivalent:** set of discrete scenarios (extreme points of polyhedron)

**special case:** \( \Gamma \)-Robustness;

\[ \mathcal{U}(\Gamma) = \left\{ a \in \mathbb{R}^n : a_i = \bar{a}_i + \hat{a}_i \delta_i, \sum_{i=1}^{n} \delta_i \leq \Gamma, \delta \in \{0, 1\}^n \right\} \]
Γ-Robust Knapsack polytope:

\[
\text{conv}\left\{ x \in \{0, 1\}^{|N|} : \sum_{i \in N} a_i \bar{a}_i x_i + \sum_{i \in S} \hat{a}_i x_i \leq b \ \forall S \subseteq N, |S| \leq \Gamma \right\}
\]

Cover inequalities for Knapsack:
Set \( C \) with \( a(C) > b \):

\[
x(C) \leq |C| - 1
\]

Extended Cover inequalities:

\[
E(C) := C \cup \{i : a_i \geq \max_{j \in C} a_j\}:
\]

\[
x(E(C)) \leq |C| - 1
\]

How to define covers for Γ-robust knapsack?
\( C \subseteq N \) is a Γ-robust cover: \( \exists S \subseteq C \) with \( |S| \leq \Gamma \) and \( \bar{a}(C) + \hat{a}(S) > b \)

What about the extension?
Scenario Extension

(\(C, S\)) a cover-pair if \(S \subseteq C\), \(|S| \leq \Gamma\), and \(\bar{a}(C) + \hat{a}(S) > b\).

Extension for cover-pair \((C, S)\):

\[
E(C, S) := C \cup \left\{ i \in N \setminus C : \bar{a}_i \geq \max_{j \in C \setminus S} \bar{a}_j, \bar{a}_i + \hat{a}_i \geq \max_{j \in S}(\bar{a}_j + \hat{a}_j) \right\}.
\]

Lemma (Büsing, K., Kutschka (2011))

\[
\sum_{j \in E(C, S)} x_j \leq |C| - 1 \text{ is a valid inequality for all cover-pairs } (C, S).
\]
Scenario Extension

\[ E(C, S) := C \cup \left\{ i \in N : \bar{a}_i \geq \max_{j \in C \setminus S} \bar{a}_j, \; \bar{a}_i + \hat{a}_i \geq \max_{j \in S}(\bar{a}_j + \hat{a}_j) \right\} . \]

- \( n = 6 \) items
- \( b = 21 \) capacity
- \( \Gamma = 2 \) robustness budget

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- \( C = \{1, 2, 3, 4\} \) robust cover
- \( S_1 = \{1, 2\} \) and \( S_2 = \{3, 4\} \) build cover-pairs with \( C = \{1, 2, 3, 4\} \)
- extensions \( E(C, S_1) = C \cup \{5\} \) and \( E(C, S_2) = C \cup \{6\} \)
- but also \( \sum_{j \in C \cup \{5,6\}} x_j \leq 3 = |C| - 1 \) is valid
- does there exist an extension \( E(C) = C \cup \{5, 6\} \)?
Union of Extensions

\[ S(C) := \{ S \subseteq C \mid (C, S) \text{ is a cover-pair} \} \] all cover-pairs with cover \( C \):

\[ E(C) := \bigcup_{S \in S(C)} E(C, S). \]

Theorem (Gersing, 2017)

Let \( C \subseteq N \) be a \( \Gamma \)-robust cover. Then

\[ \sum_{j \in E(C)} x_j \leq |C| - 1 \]

is a valid inequality for the \( \Gamma \)-robust knapsack.
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Energy System schematically

Source: ProCom
Simultaneous production of heat and power in exchange for fuel

- Fixed ratio $\rho$ between heat and power generation
- Heat can be stored for future use, power cannot be stored
- Heat storage has limited capacity and loss factor

Power has to be bought/sold at day-ahead market!
Lot-Sizing with Storage Deterioration

LS-DET:

\[
\begin{align*}
\text{min} & \quad f(q, z) + \sum_{t=1}^{T} h_t u_t \\
\text{s.t.} & \quad \alpha u_{t-1} + q_t = u_t + d_t & \forall t \in [T] \\
& \quad U_t \leq u_t \leq U_t & \forall t \in [T] \\
& \quad Qz_t \leq q_t \leq Qz_t & \forall t \in [T] \\
& \quad q_t, u_t \geq 0 & \forall t \in [T] \\
& \quad z_t \in \{0, 1\} & \forall t \in [T]
\end{align*}
\]

Lot-Sizing with

- Production limitations
- Storage limitations
- Deterioration of storage
- Concave cost function
- No backlogging

Complexity

- in general: open
- if \( Q = 0, \overline{Q} = \infty, \alpha = 1, f \) linear: \( \text{LS-DET} \in \mathcal{P} \) (Love, 1973; Atamtürk & Küçükyavuz, 2008)
- if \( U = 0, \overline{U} = \infty, \alpha = 1 \): \( \text{LS-DET} \in \mathcal{P} \) (Hellion et al., 2012)
- both cases still in \( \mathcal{P} \) if \( 0 < \alpha < 1 \) (Schmitz, 2016)

What about uncertain demands?
Forecast & Actual Heat Demands

Heat demands for week 45, 2007

Forecast error of up to 20% (average: 4.1%)
Find solutions that are feasible \textit{with high probability}!
Uncertainty Set: \( \mathcal{U} \) of possible demand realizations \((d_t)_{t \in [T]}\)

Applying Robust Optimization:

\[ \alpha u_{t-1} + q_t = u_t + d_t \quad (1b) \]

Impossible to find \((q, z, u)\) such that (1b)–(1f) are satisfied \(\forall d \in \mathcal{U}\)

**Theorem (folklore)**

Every (implicit) equality in \( Ax \leq b \) allows for the elimination of a variable involved in the equality.

\[ \Rightarrow \text{In robust optimization, elimination of variable } x \text{ implies that this variable is moved 2nd stage, i.e., after the uncertain input is known!} \]
Robust Lot-Sizing with Deterioration

RLS-DET:

\[
\begin{align*}
\text{min} & \quad f(q, z) + \eta \\
\text{s.t.} & \quad \alpha u_{t-1}(d) + q_t = u_t(d) + d_t \quad \forall t \in [T], d \in \mathcal{U} \\
& \quad U \leq u_t(d) \leq \overline{U} \quad \forall t \in [T], d \in \mathcal{U} \\
& \quad \eta \geq \sum_{t \in [T]} h^t u_t(d) \quad \forall d \in \mathcal{U} \\
& \quad Qz_t \leq q_t \leq \overline{Q}z_t \quad \forall t \in [T] \\
& \quad q_t, u_t(d) \geq 0 \quad \forall t \in [T] \\
& \quad z_t \in \{0, 1\} \quad \forall t \in [T] \\
& \quad \eta \geq 0
\end{align*}
\]

- storage \( u_t(d) \) per scenario \( d \in \mathcal{U} \)
Solving RLS-DET as LS-DET instance

**Theorem**

For an uncertainty set $\mathcal{U}$ over which a linear function can be optimized in polynomial time, RLS-DET can be **polynomially reduced** (w.r.t. production plans) to an instance of LS-DET with $d = d'$ and $\overline{U} = \overline{U}'$ thus defined:

$$d'_t := \max_{d \in \mathcal{U}} \left\{ d_t - \sum_{i=1}^{t-1} \alpha^{t-i} (d'_i - d_i) \right\} \quad \forall t \in [T] \quad (3a)$$

$$\overline{U}'_t := \overline{U}_t - \max_{d \in \mathcal{U}} \left\{ \sum_{i=1}^{t} \alpha^{t-i} (d'_i - d_i) \right\} \quad \forall t \in [T]. \quad (3b)$$
Corollary

Given an uncertainty set $\mathcal{U}$ over which a linear function can be optimized in polynomial time, RLS-DET is in $\mathcal{P}$ (resp., $\mathcal{NP}$-hard) if and only if the corresponding version of LS-DET is in $\mathcal{P}$ (resp., $\mathcal{NP}$-hard).

Robustness models satisfying precondition:

- polyhedral uncertainty sets, $\Gamma$-robustness
- discrete scenarios
- ellipsoidal uncertainty sets
Distribution of running times for $|\mathcal{U}| = 50$:

Speed-up factor between 1.82 and 85.67 with average 29.00
Outline

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4. Concluding Remarks
Capacity of optical fibre is huge, but limited!

Idea: More efficient usage of optical channels

Technology: Fixed grid vs. Flexgrid

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1 Figure taken from “Innovative Future Optical Transport Network Technologies” by T. Morioka et al., NTT Technical Review, 9 (2011).
Idea: fixed spectrum-block size → flexible block-size

- Spectrum is divided into smaller slots (e.g. 6.25GHz)
- Demands request a custom amount of these slots (‘size’)
  ⇒ Less spectrum wasted by custom-tailored slot sizes
- “Freedom” is paid for: contiguity of assigned slots required

- In future, demands will be dynamic over time
  ⇒ flexible slot allocation needed
- Question: How to allocate spectrum such that demands can “breath”? 
Definition (*Spectrum Allocation Problem* (SA))

Given a simple undirected graph $G = (V, E)$ and a set $R$ of pairs $R_i = (P_i, d_i) \in \mathcal{P} \times \mathbb{N}$, $1 \leq i \leq l$, determine

1. for every $R_i$ an interval $I_i = [a_i, b_i]$ with $a_i \leq b_i \in \mathbb{N}$ and $b_i - a_i = d_i$, such that $\max\{b_i | i = 1, \ldots, l\}$ minimal, where $I_i \cap I_j = \emptyset$ if paths $P_i$ and $P_j$ share an edge in $G$.

Let $SA(G, R)$ denote the value of an optimal solution.
Lemma (Büsing et al., 2017)

Spectrum Allocation is $\mathcal{NP}$-hard on general networks as well as on star networks

Proof for star networks: wavelength assignment ($d_i = 1$) is $\mathcal{NP}$-hard by a reduction from edge coloring.

Lemma (Büsing et al., 2017)

Spectrum Allocation is already $\mathcal{NP}$-hard on path networks and $d_i \in \{1, 2\}$

Proof: Spectrum Allocation on a path is equivalent to Dynamic Storage Allocation, which is known to be $\mathcal{NP}$-hard (GJ, 1979). Proof for $d_i \in \{c, d\}$ by Ślusarek (1987), corrected by Laube (2017).
**Theorem (Büsing et al., 2017)**

*SA is at least weakly \( \mathcal{NP} \)-hard, even if \( G \) is a path of 5 edges.*

**Proof:** Reduction from **Partition**, \( \sum_{i \in N} a_i = B \).

Note: If \( G \) is a path of \( \leq 3 \) edges, then SA can be solved in polynomial time.
Robust Spectrum Allocation: Given a number of demand scenarios $d^1, \ldots, d^K \in \mathbb{Z}^{R}$, allocate in every scenario the required number of slots such that the total number of slots across the scenarios is minimized. 
⇒ discrete uncertainty set

Applications:
- Prepare for the future: one of the $K$ scenarios will realize, but unknown which one
- Demand will fluctuate between the considered scenarios
- Multi-period Spectrum Allocation with breathing demands

Allocations can breath, but not move (service interruption):
Allocations between scenarios are interwoven!

Any Impact on Optimization?
Robust Spectrum Allocation Strategies

Five (technology) variants:

(a) RobSA-A: one joint slot

(b) RobSA-B: min. joint slots

(c) RobSA-C: nested joint slots

(d) RobSA-D: aligned (left/right)

(e) RobSA-E: overlap in central slot

Lemma

\[ RobSA_A(G, R) \leq RobSA_B(G, R) \leq RobSA_C(G, R) \leq \min\{RobSA_D(G, R), RobSA_E(G, R)\} \]
Lemma

There exists instances with $\text{RobSA}_A(G, R) < \text{RobSA}_B(G, R) < \text{RobSA}_C(G, R) < \text{RobSA}_D(G, R)$, $\text{RobSA}_C(G, R) < \text{RobSA}_E(G, R)$

Proof by example:

(a) A

(b) B

(c) C

(d) D

(e) E
Obviously: $RobSA_\ast(G, R)$ is $\mathcal{NP}$-hard to compute in general networks.

What about cases where $SA(G, R)$ is still polynomial solvable?

Polynomial solvable cases:

- $|E| = 1$, i.e., single edge case: $SA(G, R) = d(R)$
Theorem (Büsing et al., 2017)

Given a $C \in \mathbb{Z}_+$, the problems whether $\text{RobSA}_B(G, R) \leq C$ and $\text{RobSA}_C(G, R) \leq C$ are strongly NP-complete, even if $|E| = 1$ and $|K| = 2$.

Reduction from $3$-PARTITION: $3m$ items with size $a_i$, bound $B$

Define $5m$ requests with

$$d_r^k := \begin{cases} 
2a_r + 2 & \text{if } 1 \leq r \leq 3m, \ k = 1 \\
2 & \text{if } 1 \leq r \leq 3m, \ k = 2 \\
3 & \text{if } 3m + 1 \leq r \leq 5m, \ k = 1 \\
B + 3 & \text{if } 3m + 1 \leq r \leq 5m, \ k = 2 
\end{cases}$$

Corollary (Büsing et al., 2017)

Given a $C \in \mathbb{Z}_+$, the problem whether $\text{RobSA}_A(G, R) \leq C$ is strongly NP-complete, even if $|E| = 1$ and $|K| = 2$. 
Any good news?

Theorem (Büsing et al., 2017)

RobSA_D(G, R) can be solved in polynomial time on a single link.

Proof:

- Requests are aligned left or right!
- Slots can be saved by combining a left and right request
- Min. weighted perfect matching on complete graph K_{|R|} has to be solved

What about E?
**Theorem (Büsing et al., 2017)**

Let $|K| = 2$ and let $d_r^k$ be odd for all $r \in R$ and $k \in K$. Then, $\text{RobSA}_E(G, R)$ on a single link is polynomial-time solvable.

**Proof:** $\text{RobSA}_E$ can be modelled as Gilmore-Gomory-TSP: NP-complete cases of variants D and E?

**Theorem (Büsing et al., 2017)**

Given a $C \in \mathbb{Z}_+$, the problem whether $\text{RobSA}_D(G, R) \leq C$ is strongly NP-complete, even if $|E| = 2$ and $|K| = 2$.

Reduction from **3-PARTITION**

**Theorem (Büsing et al., 2017)**

Given a $C \in \mathbb{Z}_+$, the problem whether $\text{RobSA}_E(G, R) \leq C$ is strongly NP-complete, even if $|E| = 1$ and $|K| = |R|$ or $|E| = 2$ and $|K| = 2$.

Reductions from **HAMILTONIAN PATH** and **3-PARTITION**, respectively.
Without uncertainty:

| Graph $G$ | $d_r = c$ | $d_r \in \{c, d\}$ | $|P_r| \leq k$, $k \geq 3$ | $|P_r| = 3$ | $|P_r| \leq 2$ |
|-----------|------------|------------------|-------------------|-----------|----------------|
| $S_{1,n}$ | str. NP-c  | str. NP-c        | str. NP-c         | -         | str. NP-c      |
| $P_n$     | $\mathcal{P}$ | str. NP-c   | weak NP-c        | weak NP-c | $\mathcal{P}$ |
| $P_n$, $n = 6$ | $\mathcal{P}$ | open          | weak NP-c        | $\mathcal{P}$ | $\mathcal{P}$ |
| $P_n$, $n = 5$ | $\mathcal{P}$ | open          | open            | $\mathcal{P}$ | $\mathcal{P}$ |
| $P_n$, $n \leq 4$ | $\mathcal{P}$ | $\mathcal{P}$ | $\mathcal{P}$    | $\mathcal{P}$ | $\mathcal{P}$ |

With uncertainty:

| Graph $G$ | $|K| = 2$ | $|K| = |R|$ | general |
|-----------|-----------|-----------|---------|
| $|E| = 1$  | str. NP-c | $\mathcal{P}$ | str. NP-c |
| $|E| \geq 2$ | str. NP-c | str. NP-c | str. NP-c |
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Incorporation of Uncertainties in Optimization pays off!

- ProCom @E-world 2017: BoFiT Optimierung 7.0 – Robust Optimization

but impacts solution process

Different ways to model uncertainties yield different results:

- Multi-Stage Robustness, Recoverable Robustness, Chance-Constrained Models, Affine Models, etc.
- Evaluation determines feasibility of approach

New theory:

- Robust valid inequalities for knapsack, network design, etc.
- Robust Lot-Sizing can be solved as deterministic Lot-Sizing
- Complexity border yields useful insights on robust concepts

Optimization under Uncertainties: just do it!
Combinatorial Optimization inspired by Uncertainties

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