**Lehrstuhl II für Mathematik** http://www.math2.rwth-aachen.de
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# 17 Differential Equations

## Introduction

A differential equation is an equation which contains an unknown function and one of its unknown derivatives. The Goal is to find the unknown function. A simple first order differential equation has the form $y' = f(y, x)$.

**Examples:**

i) Population growth can be represented with an differential equation:

$$y'(x) = \alpha\, y(x),\ \alpha \in \mathbb{R}$$

Let be $y(0) = y_0$ the population at time $x = 0$. The solution of this equation is

$$y(x) = y_0 e^{\alpha x}.$$

Test:

$$y(0) = y_0 e^0 = y_0 \quad \checkmark$$
$$y'(x) = \alpha y_0 e^{\alpha x} = \alpha y(x) \quad \checkmark$$

ii) Newtons Law

$$\begin{aligned} y(t) &: \text{ position of a mass point at time } t \\ y(0) = y_0 &: \text{ position of a mass point at time } 0 \\ v_0 &: \text{ initial velocity} \\ y''(t) = -g &: \text{ acceleration due to gravity} \end{aligned}$$

The solution is the free fall law

$$y(t) = -\frac{1}{2} g t^2 + v_0 t + y_0$$

**Definition 1** (Ordinary Differential Equation)**.** *Let $F \colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a function. Then*

$$F(x, y, y', y'', \dots, y^{(n-1)}) = y^{(n)}$$

*is called an* ordinary differential equation (ODE) *of order $n$. An $n$-times continuously differentiable function $y \colon I \to \mathbb{R}$ is called* solution *if it fulfills the ODE for all $x \in I$.*

**Definition 2** (exact differential equations). *Let two differentiable functions $P : \mathbb{R}^2 \to \mathbb{R}$ and $Q : \mathbb{R}^2 \to \mathbb{R}$ with continuous derivatives be given. A differential equation of the form*

$$P(x, y)\, dx + Q(x, y)\, dy = 0$$

*is said to be* exact, *if*

$$\frac{\partial P(x, y)}{\partial y} = \frac{\partial Q(x, y)}{\partial x}.$$

*Solution: There is a function $F(x, y)$ with $F_x = P$ and $F_y = Q$.*

$$\Rightarrow \quad F(x, y) = c, \;\; c \in \mathbb{R} \;\; \textit{is the implicit solution.}$$

Integrating factors:
*Some (non-exact) differential equations of the form $P(x, y)dx + Q(x, y)dy = 0$ can be made exact by multiplying them with a suitable function $M(x, y)$, the so-called integrating factor.*

$$M(x, y)P(x, y)\, dx + M(x, y)Q(x, y)\, dy = 0$$

**Theorem 3** (Separation of variables). *Suppose a first order ODE can be written in the form*

$$\frac{dy}{dt} = g(x)h(y).$$

*Obviously, all constant functions $y = c$ with $h(y) = 0$ are solutions of the ODE. If $h(y) \neq 0$, the terms can be re-arranged to*

$$\int \frac{1}{h(y)} dy = \int g(x) dx.$$

*Solve this equation for $y$ to compute the remaining solutions of the ODE.*

**Definition 4** (Linear Differential Equation). *A* linear differential equation (LDE) *of order $n$ is an ODE of the form*

$$y^{(n)} + A_1(x)y^{(n-1)} + A_2(x)y^{(n-2)} + \cdots + A_n(x)y = f(x),$$

*where $A_i, f : \mathbb{R} \to \mathbb{R}$ are functions. If $f = 0$, the LDE is called homogenous, and inhomogenous otherwise.*

**Theorem 5.** *Let*

$$y'(x) + a(x)y(x) = 0$$

*be a homogenous first order LDE. Then its set of solutions is given by*

$$\left\{ y(x) = ce^{-A(x)}, c \in \mathbb{R} \right\},$$

*where $A$ is a primitive of $a$. ($A(x) = \int a(x)\, dx$)*

**Theorem 6** (Variation of constant)**.** *Let*

$$y'(x) + a(x)y(x) = f(x)$$

*be an inhomogenous first order LDE. Then its set of solutions is given by*

$$\left\{ e^{-A(x)} \left( c + \int f(x) e^{A(x)} dx \right), c \in \mathbb{R} \right\},$$

*where $A$ is a primitive of $a$.*

$$y(x) = e^{-A(x)} \left( y_0 + \int_{x_0}^{x} f(t)\, e^{A(t)}\, dt \right)$$

*is a solution of the differential equation satisfying $y(x_0) = y_0$.*

**Definition 7** (Bernoulli differential equations)**.** *A differential equation of the form*

$$y' + g(x)y = h(x)y^{\alpha}, \;\; \alpha \in \mathbb{R} \setminus \{0, 1\},$$

*with functions $g, h : \mathbb{R} \to \mathbb{R}$ is called* Bernoulli differential equation. *Substitution: $z = y^{1-\alpha}$*

$$\Rightarrow \quad z' + (1-\alpha)g(x)z = (1-\alpha)h(x)$$

*This is a linear differential equation in $z$ and can be solved with the formulas above.*

**Definition 8** (Riccati differential equations)**.** *A differential equation of the form*

$$y' + g(x)y + h(x)y^2 = f(x)$$

*with functions $f, g, h : \mathbb{R} \to \mathbb{R}$ is called* Riccati differential equation. *$y_s$ shall be a special solution of this differential equation.*
*Substitution: $z = \dfrac{1}{y - y_s}$*

$$\Rightarrow \quad z' - [g(x) + 2h(x)y_s]z = h(x)$$

*This is a linear differential equation in $z$ and can be solved with the formulas above. Back-substituting yields $y = y_s + \dfrac{1}{z}$.*

**Definition 9** (matrix differential equation)**.** *Let*

$$\mathbf{A} := \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \quad i, j = 1 \ldots, n$$

*an $n \times n$ matrix, of which all elements are constants. Then*

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t), \quad \mathbf{y} = (y_1, \ldots, y_n)$$

*is called a first order matrix differential equation (MDE)*

**Theorem 10.** *Let* $\mathbf{y}' = A\mathbf{y}$ *be a first order MDE, where* $A$ *is an* $(n \times n)$*-matrix with real entries. If* $A$ *has* $n$ *different eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *with corresponding eigenvectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$*, then the solution set of the MDE is given by*

$$\left\{ \mathbf{y}(t) = \sum_{k=1}^{n} a_k \mathbf{v}_k e^{\lambda_k t} : a_k \in \mathbb{R} \right\}.$$

*(If initial conditions are given, the coefficient* $a_k$ *may be computed accordingly.) For an arbitrary matrix* $A$*, the solution set can be computed by using its generalized eigenvectors.*