15 Integration

Definition 1 (supremum, infimum). *Let M be a subset of $\mathbb{R}$. If M is bounded from above, the smallest upper bound of M is called* supremum *denoted by*

$$\sup M = \min\{a \in \mathbb{R} \colon x \leq a \text{ for all } x \in M\}.$$

If M is bounded from below, the greatest lower bound of M is called infimum *denoted by*

$$\inf M = \max\{b \in \mathbb{R} \colon x \geq b \text{ for all } x \in M\}.$$

Definition 2 (partition). *A partition of an interval $[a, b]$ is given by a finite subset $\mathcal{P} = \{x_0, x_1, \ldots, x_r\}$ of $[a, b]$ satisfying $\{a, b\} \subset \mathcal{P}$. We write $a = x_0 < x_1 < \cdots < x_r = b$.*

Definition 3. *Let $f \colon [a, b] \to \mathbb{R}$ be a function and $\mathcal{P}$ a partition of $[a, b]$ such that $a = x_0 < x_1 < \cdots < x_r = b$. For $i = 0, 1, \ldots, r-1$ we define*

$$M_i^{\mathcal{P}}(f) = \sup\{f(x) \colon x_i < x < x_{i+1}\},$$
$$m_i^{\mathcal{P}}(f) = \inf\{f(x) \colon x_i < x < x_{i+1}\}.$$

Definition 4 (upper/lower sum). *Let $f \colon [a, b] \to \mathbb{R}$ be a function and $\mathcal{P}$ a partition of $[a, b]$ such that $a = x_0 < x_1 < \cdots < x_r = b$. The* upper sum *of f over $\mathcal{P}$ is*

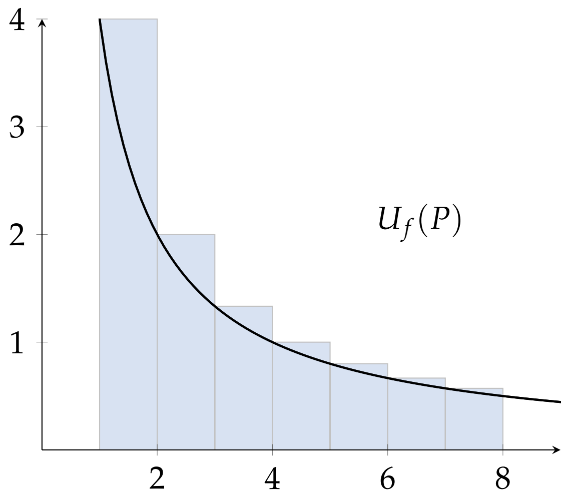
$$U_f(P) = \sum_{k=1}^{r} M_{k-1}^{\mathcal{P}}(f)(x_k - x_{k-1}),$$

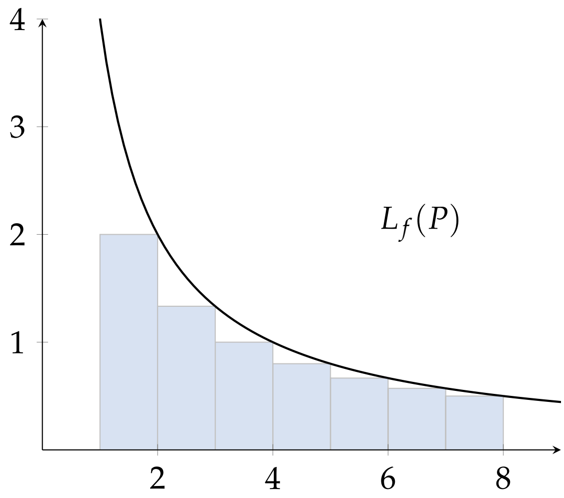
and the lower sum *of f over $\mathcal{P}$*

$$L_f(P) = \sum_{k=1}^{r} m_{k-1}^{\mathcal{P}}(f)(x_k - x_{k-1}).$$

Definition 5 (Riemann integrable functions). *A function $f \colon [a, b] \to \mathbb{R}$ is called* (Riemann) integrable *if for every $\varepsilon > 0$ there is a partition $\mathcal{P}$ of $[a, b]$ such that*

$$U_f(\mathcal{P}) - L_f(\mathcal{P}) < \varepsilon.$$





Definition 6. *Let f a* RIEMANN *integrable function and let $\mathcal{P}$ a partition of $[a,b]$. The limit*

$$I(f) = \lim_{r\to\infty} U_f(\mathcal{P}) = \lim_{r\to\infty} L_f(\mathcal{P})$$

is called the (RIEMANN) integral *of f over $[a,b]$ and we write*

$$I(f) = \int_a^b f(x)\,\mathrm{d}x$$

Definition 7. *The function f is the* integrand*. For $a < b$ we define*

$$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$$

and for $a = b$ we define

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^a f(x)\,\mathrm{d}x = 0.$$

Theorem 8. *If f is continuous on $[a,b]$, then f is integrable.*

Theorem 9. *If f, g are integrable over $[a,b]$ and $r \in \mathbb{R}$, then*

(i) *rf is integrable over $[a,b]$ with* $\int_a^b (rf)(x)\,\mathrm{d}x = r\int_a^b f(x)\,\mathrm{d}x.$

(ii) *$f+g$ is integrable over $[a,b]$ with* $\int_a^b (f+g)(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x.$

(iii) *for $a < c < b$:* $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$

(iv) *$|f|$ is integrable on $[a,b]$ with* $|\int_a^b f(x)\,\mathrm{d}x| \le \int_a^b |f(x)|\,\mathrm{d}x.$

Theorem 10. *Let f be integrable over $[a, b]$ and bounded with $m \leq f(x) \leq M$ for all $x \in [a, b]$. Then*

$$m(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a)$$

and, in particular, if f is continuous, then

$$(b-a) \min_{x\in[a,b]} f(x) \leq \int_a^b f(x)\,\mathrm{d}x \leq (b-a) \max_{x\in[a,b]} f(x).$$

More generally, if f, g are integrable over $[a, b]$ with $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x.$$

Theorem 11 (Mean Value Theorem for Integration)**.** *Let f be continuous over $[a, b]$. Then there exists a $\xi \in [a, b]$ such that*

$$\int_a^b f(x)\,dx = f(\xi)(b-a).$$

Theorem 12 (Fundamental Theorem of Calculus)**.** *Let $f\colon [a, b] \to \mathbb{R}$ be continuous. Then*

$$F : [a, b] \to \mathbb{R}, \text{ with } F(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$, and differentiable on (a, b) with

$$F'(x) = f(x) \text{ for all } x \in (a, b).$$

Conversely, if $F\colon [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b) with $F' = f$, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = F(x)|_a^b.$$

Definition 13 (primitive)**.** *If $f\colon [a, b] \to \mathbb{R}$ is continuous, then a differentiable function F with $F' = f$ is called* primitive *of f. (*One *primitive of f is given by $F(x) = \int_a^x f(x)\,\mathrm{d}x$. The difference of two primitives of the same function is constant.)*

Theorem 14 (Partial integration)**.** *Let $f, g\colon [a, b] \to \mathbb{R}$ such that f is continuous, g is differentiable, and g' is continuous. If F is a primitive of f, then*

$$\int_a^b f(x)g(x)\,\mathrm{d}x = F(x)g(x)|_a^b - \int_a^b F(x)g'(x)\,\mathrm{d}x.$$

Theorem 15 (Substitution rule)**.** *Let $f\colon [c, d] \to \mathbb{R}$ be continuous, and $g\colon [a, b] \to \mathbb{R}$ be differentiable such that g' is continuous. If $g([a, b]) \subset [c, d]$, then*

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(t)dt.$$

Theorem 16 (Partial Fraction Decomposition). *Given two polynomials $p_m(x)$ and $q_n(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ and degree $m < n$, partial fraction are generally obtained by supposing that*

$$\frac{p_m(x)}{q_n(x)} = \frac{A_1}{(x-\alpha_1)} + \frac{A_2}{(x-\alpha_1)} + \cdots + \frac{A_n}{(x-\alpha_n)}$$

and solving the constants $A_i \in \mathbb{R}$ by equating the coefficients. We get

$$\begin{aligned} \int \frac{p_m(x)}{q_n(x)}\,dx &= \int \frac{A_1}{(x-\alpha_1)} + \frac{A_2}{(x-\alpha_1)} + \cdots + \frac{A_n}{(x-\alpha_n)}\,dx \\ &= A_1 \ln(x-\alpha_1) + A_2 \ln(x-\alpha_2) + \cdots + A_n \ln(x-\alpha_n). \end{aligned}$$

Primitives of elementary functions		
$\int x^n\,dx = \frac{1}{n+1}x^{n+1}$ $(n\neq -1)$,	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert$,	$\int \sqrt[n]{x}\,dx = \frac{n}{n+1}(\sqrt[n]{x})^{n+1}$
$\int \sin x\,dx = -\cos x$,	$\int \cos x\,dx = \sin x$,	$\int \tan x\,dx = -\ln\lvert\cos x\rvert$
$\int \cot x\,dx = \ln\lvert\sin x\rvert$,	$\int \frac{1}{\cos^2 x}\,dx = \tan x$,	$\int \frac{1}{\sin^2 x}\,dx = -\cot x$
$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x$,	$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos x$,	$\int \frac{1}{1+x^2}\,dx = \arctan x$
$\int \sinh x\,dx = \cosh x$,	$\int \cosh x\,dx = \sinh x$,	$\int \tanh x\,dx = \ln(\cosh x)$
$\int \frac{1}{\sqrt{1+x^2}}\,dx = \operatorname{arsinh} x$,	$\int \frac{1}{\sqrt{x^2-1}}\,dx = \operatorname{arcosh} x$,	$\int \frac{1}{1-x^2}\,dx = \operatorname{artanh} x$ $(\lvert x\rvert<1)$
$\int e^x\,dx = e^x$,	$\int a^x\,dx = \frac{1}{\ln a}a^x$ $(a>0)$,	$\int \ln x\,dx = x\ln x - x$
$\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert$,	$\int f'(x)\cdot f(x)\,dx = \frac{1}{2}(f(x))^2$	