Lehrstuhl II für Mathematik http://www.math2.rwth-aachen.de
Dipl.-Math. Michael Hoschek **WiSe 2013/2014**

14 Special Functions

Definition 1 (exponential function)**.** *The* exponential function *is defined by*

$$\exp\colon \mathbb{R} \to \mathbb{R}$$
$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n.$$

Theorem 2. *The exponential function has the following properties.*

(i) exp *is continuous on* $\mathbb{R}$*;*

(ii) $\exp(x) > 0$ *for all* $x \in \mathbb{R}$*;*

(iii) for all $x, y \in \mathbb{R}$*:* $\exp(x + y) = \exp(x) \cdot \exp(y)$*;*

(iv) exp *is indefinitely differentiable and* $\exp' = \exp$*;*

(v) exp *is strictly increasing on* $\mathbb{R}$*;*

(vi) $\lim_{x\to-\infty} \exp(x) = 0$ *and* $\lim_{x\to\infty} \exp(x) = \infty$*;*

(vii) $\exp\colon \mathbb{R} \to (0, \infty)$ *is bijective;*

(viii) $\exp(1) = e$ *and* $\exp(\frac{n}{m}) = e^{n/m}$ *for* $n, m \in \mathbb{Z}$ *and* $m \neq 0$*.*

Definition 3. *For* $x \in \mathbb{R}$ *we define* $e^x = \exp(x)$*.*

Definition 4 (natural logarithm)**.** *The* (natural) logarithm $\ln\colon (0, \infty) \to \mathbb{R}$ *is the inverse function of* exp*.*

Theorem 5. *The natural logarithm has the following properties.*

(i) ln *is continuous on* $(0, \infty)$*;*

(ii) $\ln(x) < 0$ *for* $x \in (0, 1)$*,* $\ln(1) = 0$ *and* $\ln(x) > 0$ *for* $x \in (1, \infty)$*;*

(iii) for all $x, y \in (0, \infty)$*:* $\ln(xy) = \ln(x) + \ln(y)$*;*

(iv) ln *is differentiable with* $\ln'(x) = \frac{1}{x}$*;*

(v) ln *is strictly increasing on* $(0, \infty)$*;*

(vi) $\lim_{x\to 0} \ln(x) = -\infty$ *and* $\lim_{x\to\infty} \ln(x) = \infty$*;*

(vii) $\ln\colon (0,\infty) \to \mathbb{R}$ *is bijective.*

Definition 6 (arbitrary powers)**.** *For* $a, x \in \mathbb{R}$ *with* $a > 0$ *we define* $a^x = e^{x\ln(a)}$.

Theorem 7. *The function* $f(x) = a^x$ *has the following properties.*

(i) $f\colon \mathbb{R} \to (0,\infty)$ *is bijective;*

(ii) for all $x, y \in \mathbb{R}$*:* $f(x+y) = a^{x+y} = a^x a^y = f(x)f(y)$*;*

(iii) for all $x, y \in \mathbb{R}$*:* $f(xy) = a^{xy} = (a^x)^y = f(x)^y$*;*

(iv) f *is differentiable with* $f'(x) = f(x)\ln(x)$.

Definition 8 (arbitrary logarithm)**.** *The inverse function of* $f(x) = a^x$ *is called* base a logarithm*, denoted by* $\log_a$.

Theorem 9. *The base a logarithm can be computed as*

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

Theorem 10. *The base a logarithm is differentiable with*

$$(\log_a(x))' = \frac{1}{x\ln(a)}.$$

Important functions

$f(x)$	$\sum a_k x^k$	$f(x)$	$\sum a_k x^k$
$\exp x$	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$	$\ln(1+x)$	$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}, \lvert x\rvert \le 1,\ x \neq -1$
$\cos x$	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$	$\cosh x$	$\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$
$\sin x$	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$	$\sinh x$	$\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$