13 Applications of Differentiability

Definition 1 (minimum/maximum point). *Let $f\colon M \to \mathbb{R}$, where $M \subset \mathbb{R}^n$. Then $x_0 \in M$ is a*

(i) local minimum [maximum] point *if there exists a suitable $\delta > 0$ such that $f(x_0) \leq f(x)$ [$\geq f(x)$] for all $x \in M$ with $|x_0 - x| < \delta$;*

(ii) global minimum [maximum] point *if $f(x_0) \leq f(x)$ [$\geq f(x)$] for all $x \in M$.*

Theorem 2. *Let $f\colon (a,b) \to \mathbb{R}$ be differentiable. If $x_0 \in (a,b)$ is a local minimum or maximum point, then $f'(x_0) = 0$.*

Theorem 3 (Extremal Test). *Let $f\colon (a,b) \to \mathbb{R}$ be 2 times differentiable and $x_0 \in (a,b)$ with $f'(x_0)$.*

$$\begin{aligned} f''(x_0) > 0 \quad &\Rightarrow \quad x_0 \text{ is a local minimum} \\ f''(x_0) < 0 \quad &\Rightarrow \quad x_0 \text{ is a local maximum} \end{aligned}$$

Definition 4. *Let $f\colon (a,b) \to \mathbb{R}$ be differentiable. A point x_0 is an* inflection point *if*

$$\begin{aligned} f''(x_0) &= 0 \\ f'''(x_0) &\neq 0. \end{aligned}$$

Theorem 5 (higher order derivative test). *Let $f\colon [a,b] \to \mathbb{R}$ be n-times differentiable on (a,b) for some $n \in \mathbb{N}$. Let $x_0 \in (a,b)$ such that*

$$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$$

and $f^{(n)}(x_0) \neq 0$.

(i) If n is even and $f^{(n)}(x_0) < 0$ [> 0], then x_0 is a local maximum [minimum] point.

(ii) If n is odd, then x_0 is an inflection point.

Theorem 6 (Rolle's theorem). *Let $f\colon [a,b] \to \mathbb{R}$ be continuous, and differentiable on (a,b). If $f(a) = f(b)$, then there exists $\xi \in (a,b)$ with $f'(\xi) = 0$.*

Corollary 7 (Mean Value Theorem). *Let $f\colon [a,b] \to \mathbb{R}$ be continuous, and differentiable on (a,b). Then there exists $\xi \in (a,b)$ with*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Corollary 8. *If $f\colon (a,b) \to \mathbb{R}$ is differentiable with $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.*

Theorem 9. *If $f\colon (a,b) \to \mathbb{R}$ is differentiable, then*

(i) f is [strictly] increasing if and only if $f'(x) \geq 0$ [> 0] for all $x \in (a,b)$;

(ii) f is [strictly] decreasing if and only if $f'(x) \leq 0$ [< 0] for all $x \in (a,b)$.

Theorem 10. *Let $f\colon [a,b] \to \mathbb{R}$ and $x_0 \in [a,b]$. If f is decreasing [increasing] in $(x_0 - \delta, x_0] \cap [a,b]$ and increasing [decreasing] in $[x_0, x_0 + \delta) \cap [a,b]$ for some $\delta > 0$, then x_0 is a local minimum [maximum] point.*

Theorem 11 (De l'Hospital's Theorem). *Let $f, g\colon (a,b) \to \mathbb{R}$ be differentiable and $x_0 \in [a,b]$. Let either*

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$$

or

$$|\lim_{x \to x_0} f(x)| = |\lim_{x \to x_0} g(x)| = \infty.$$

If

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} \to y \in \mathbb{R},$$

then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} \to y \in \mathbb{R}.$$

(An analogue holds for the case that '$x_0 = \pm\infty$'.)

Definition 12 (Taylor polynomial). *Let $f\colon D \to \mathbb{R}$ be n times differentiable, where $D \subset \mathbb{R}$ and $n \in \mathbb{N}$. For $x_0 \in D$, the* Taylor polynomial of f of degree n at x_0 *is defined as*

$$\begin{aligned} T^n_{x_0}(f)(x) &= T^n_{x_0} f(x) \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \end{aligned}$$

Definition 13 (TAYLOR series)**.** *If $f\colon D \to \mathbb{R}$, where $D \subset \mathbb{R}$, is an indefinitely differentiable function and $x_0 \in D$, then*

$$T_{x_0}^{\infty} f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

is the TAYLOR series of f at x_0.

Theorem 14 (TAYLOR's theorem)**.** *Let $f\colon D \to \mathbb{R}$ be n times differentiable, where $D \subset \mathbb{R}$ and $n \in \mathbb{N}$. If $x_0 \in D$, then*

$$f(x) = T_{x_0}^{n} f(x) + R_{x_0}^{n+1} f(x)$$

with

$$R_{x_0}^{n+1} f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

for a number ξ between x and x_0, i.e.

$$\xi \in \begin{cases} [x, x_0] & \text{if } x < x_0 \\ [x_0, x] & \text{if } x > x_0 \end{cases}.$$

The summand $R_{x_0}^{n+1} f(x)$ is called error or remainder term.