Lehrstuhl II für Mathematik http://www.math2.rwth-aachen.de
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12 Differentiability

Definition 1 (secant, slope & difference quotient). *Let (a, b) be an open interval and $x_0 \in (a, b)$. Let $f\colon (a, b) \to \mathbb{R}$ be a function.*

(i) The secant *to f through x_0 and x is the straight line connecting $(x_0, f(x_0))$ and $(x, f(x))$.*

(ii) The slope *of the secant through x_0 and x given as*

$$\triangle f(x_0; x) = \frac{f(x) - f(x_0)}{x - x_0}$$

is called the difference quotient.

Definition 2 (differentiability & derivative). *Let $D \subset \mathbb{R}$ and $f\colon D \to \mathbb{R}$ be a function. Then f is called* differentiable at $x_0 \in D$ *if for some $\delta > 0$: $(x_0 - \delta, x_0 + \delta) \subset D$ and*

$$f'(x_0) = \lim_{x \to x_0} \triangle f(x_0; x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists in $\mathbb{R}$. The limit $f'(x_0)$ is called derivative *of f at x_0. If f is differentiable at every point $x_0 \in D$, the function $f'\colon D \to \mathbb{R}$ is called* derivative (function) *of f.*

There is another form of the difference quotient. We replace $x - x_0$ by h and consider

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0).$$

Theorem 3 (differentiability & continuity). *If f is differentiable, it is continuous.*

Theorem 4 (differentiability & operations on functions). *Let $f, g\colon D \to \mathbb{R}$ be differentiable functions, where $D \subset \mathbb{R}$. Then*

(i) $(f + g)' = f' + g'$,

(ii) $(fg)' = f'g + fg'$,

(iii) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ *whenever $g \neq 0$.*

Theorem 5 (chain rule). *Let $f\colon D_1 \to D_2$ and $g\colon D_2 \to \mathbb{R}$, where $D_1, D_2 \subset \mathbb{R}$, be continuous functions. If f is differentiable in x_0 and g is differentiable in $f(x_0)$, then $g \circ f\colon D_1 \to \mathbb{R}$ is differentiable in x_0 and*

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

Theorem 6 (differentiability of f^{-1}). *Let $f\colon I \to \mathbb{R}$ be a differentiable and strictly monotonic function, where $I \subset \mathbb{R}$ is an interval. Then $f^{-1}\colon f(I) \to I$ is differentiable and*

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}.$$

Definition 7 (higher order derivatives). *Let $f\colon D \to \mathbb{R}$, where $D \subset \mathbb{R}$, be a differentiable function. If f' is differentiable on D, the derivative of f' is called* second derivative *of f:*

$$f^{(2)} = f''.$$

In this case f is called twice differentiable. *More generally, if f is n-times differentiable, $n \in \mathbb{N}$, and its n-th derivative $f^{(n)}$ is differentiable, the* $(n+1)$-st derivative *of f is the derivative of $f^{(n)}$:*

$$f^{(n+1)} = (f^{(n)})'.$$

In this case f is called $(n+1)$-times differentiable. *If $f^{(n)}$ exists for all $n \in \mathbb{N}$, then f is* indefinitely differentiable.

An arbitrary function with domain $\mathbb{R}^n$ and target $\mathbb{R}^m$ has the form $f(x_1, \ldots, x_n) = (f_1, \ldots, f_m)$ where each of the component functions, f_i , is a real-valued function with domain $\mathbb{R}^n$. To make the dependence on n variables explicit, we can write $f_i(x_1, \ldots, x_n)$.

Definition 8 (partial derivative). *A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is called* partially differentiable at $\mathbf{x}_0$ in direction x_i *if the limit*

$$\frac{\partial f}{\partial x_i}(\mathbf{x}_0) = \partial_j f(\mathbf{x}_0) = \lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{e}_i) - f(\mathbf{x}_0)}{t} \in \mathbb{R}^m$$

exists. Here $\mathbf{e}_i$ is the i-th unit vector. The limit $\partial_j f(\mathbf{x}_0)$ is called i-th partial derivative *of f at x_0. If f is partially differentiable at every point $x_0 \in D$ in every direction x_i, then f is called* partially differentiable on D.

Definition 9 (directional derivative). *A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is called* directionally differentiable at $\mathbf{x}_0$ along $\mathbf{v} \neq \mathbf{0}$ *if the limit*

$$\partial_{\mathbf{v}} f(\mathbf{x}_0) = \lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{v}) - f(\mathbf{x}_0)}{t} \in \mathbb{R}^m$$

exists. The limit $\partial_{\mathbf{v}} f(\mathbf{x}_0)$ is called directional derivative of f at $\mathbf{x}_0$ along $\mathbf{v}$.

Definition 10 (Jacobian Matrix)**.** *The* JACOBIAN *matrix for* $f = f(x_1, \dots, x_n) = (f_1, \dots, f_m)$ *is a matrix filled with partial derivatives. The entry in the i-th row and j-th column is* $\frac{\partial f_i}{\partial x_j}$*:*

$$J_f := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Theorem 11. *If* $f\colon D \to \mathbb{R}$ *is twice continuously differentiable at* $\mathbf{x}_0 \in D$*, then*

$$\partial_i \partial_j f(\mathbf{x}_0) = \partial_j \partial_i f(\mathbf{x}_0).$$

Definition 12. *If* $f\colon D \to \mathbb{R}$ *is twice continuously differentiable at* $\mathbf{x}_0 \in D$*, then the matrix* $Hf(\mathbf{x}_0)$ *defined by*

$$(Hf(\mathbf{x}_0))_{i,j} = \partial_i \partial_j f(\mathbf{x}_0)$$

for $i, j = 1, 2, \dots, n$ *is called the* HESSIAN matrix of f at $\mathbf{x}_0$*, or simply* HESSIAN*. (With regard to Theorem 11, the* HESSIAN *is symmetrical.)*

Derivatives of elementary functions

$f(x)$	$f'(x)$
$c = const$	0
x^n	nx^{n-1}
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$\frac{-1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
e^x	e^x
a^x	$\ln(a)a^x$
$\ln(x)$	$\frac{1}{x}$