Lehrstuhl II für Mathematik http://www.math2.rwth-aachen.de
Dipl.-Math. Michael Hoschek **WiSe 2013/2014**

10 Series

Definition 1 (series)**.** *Let* (x_n) *be a sequence. The* series $\sum_{n=0}^{\infty} x_n$ *is the sequence* (y_n) *of* partial sums

$$y_n = \sum_{k=0}^{n} x_k.$$

The series converges to $y \in \mathbb{R}$ *if* $y_n \to y$*. In this case we write*

$$\sum_{n=0}^{\infty} x_n = y.$$

A series converges absolutely *if*

$$\sum_{n=0}^{\infty} |x_n|$$

converges. We also use the notation $\sum x_n$*.*

Theorem 2. *If* $\sum\limits_{n=0}^{\infty} x_n$ *converges, then* $x_n \to 0$*.*

Theorem 3 (majorant criterion)**.** *Let* $x_n, y_n \geq 0$ *for* $n \in \mathbb{N}_0$*. If there exists a number* $n_0 \in \mathbb{N}_0$ *such that*

$$\text{for all } n \geq n_0 \colon x_n \leq y_n,$$

then the following holds.

(i) If $\sum y_n$ *converges, then* $\sum x_n$ *converges.*

(ii) If $\sum x_n$ *diverges, then* $\sum y_n$ *diverges.*

Proposition 4 ((absolute) convergence)**.** *If a series converges absolutely, then it is convergent.*

Theorem 5 (root criterion)**.** *Let* $x_n \geq 0$ *for* $n \in \mathbb{N}_0$*. If*

$$\lim_{n\to\infty} \sqrt[n]{x_n} < 1,$$

then $\sum x_n$ *is convergent. If*

$$\lim_{n\to\infty} \sqrt[n]{x_n} > 1,$$

then $\sum x_n$ *is divergent.*

Theorem 6 (quotient criterion). *Let $x_n > 0$ for $n \in \mathbb{N}_0$. If*

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} < 1,$$

then $\sum x_n$ is convergent. If

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} > 1,$$

then $\sum x_n$ is divergent.

Theorem 7 (Leibniz criterion). *If (x_n) is a monotonically decreasing sequence with $x_n \to 0$, then $\sum(-1)^n x_n$ converges.*

Definition 8 (power series). *For $a_k \in \mathbb{R}$, where $k \in \mathbb{N}_0$, and $x_0 \in \mathbb{R}$, a series*

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k$$

is called a power series.

Theorem 9 (radius of convergence). *Let $\sum_{k=0}^{\infty} a_k (x-x_0)^k$ be a power series. If*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = c,$$

the power series converges for every x with $|x - x_0| < \frac{1}{c}$ and diverges for every x with $|x - x_0| > \frac{1}{c}$. (If $c = 0$, the power series converges for every x.) If

$$\lim_{n\to\infty} |\frac{a_n}{a_{n+1}}| = r,$$

the power series converges for every x with $|x - x_0| < r$ and diverges for every x with $|x - x_0| > r$. Moreover, if one of the two limits exists, the other exists too, and $\frac{1}{c} = r$. The number r is called the radius of convergence *and $(x_0 - r, x_0 + r)$ is called the* interval of convergence.