8 Mappings

Definition 1 (intuitive definition of a mapping)**.** *Let M, N be two sets. A* mapping *$f \colon M \to N$ is a rule that assigns each $x \in M$ a unique element $y = f(x) \in N$. The set M is called domain, the set N is called* codomain *or* target set*. Two mappings f, g are* equal *if their domains and codomains are equal, and $f(x) = g(x)$ for every $x \in D(f)$.*

Definition 2 (function)**.** *A real-valued mapping, i.e. a mapping $f \colon M \to \mathbb{R}$, is called a* function*.*

Definition 3 (domain & range)**.** *If $f \colon M \to N$ is a mapping, $A \subset M$ and $B \subset N$, then*

$$f(A) = \{f(x) \colon x \in A\} \subset N,$$
$$f^{-1}(B) = \{x \in M \colon f(x) \in B\} \subset M$$

are the image *of A and the* preimage *of B. The set $f(M)$ is also called* range *of f.*

Definition 4 (graph)**.** *Let $f \colon M \to N$ be a mapping. The set*

$$G_f = \{(x, f(x)) \colon x \in M\} \subset M \times N$$

is the graph *of f.*

Definition 5 (formal definition of a mapping)**.** *A* mapping *$f \colon M \to N$ is a subset $f \subset M \times N$ with the property that for each $x \in M$ there is precisely one $y \in N$ such that $(x, y) \in f$.*

Definition 6 (maximal domain)**.** *Given an expression $y = f(x)$, the* maximal domain of definition *of (the function) f is the set*

$$D(f) = \{x \in \mathbb{R} \colon f(x) \text{ is well-defined}\}.$$

Definition 7 (operations on functions). *Let $f, g\colon M \to \mathbb{R}$ be functions. We define*

- $(f+g)(x) = f(x) + g(x)$,
- *for* $\lambda \in \mathbb{R}$: $(\lambda f)(x) = \lambda f(x)$,
- $(f \cdot g)(x) = f(x)g(x)$,
- *for* $g(x) \neq 0$: $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$.

Definition 8 (composition of mappings). *If $f\colon M \to N$ and $g\colon N \to U$ are mappings, then*

$$(g \circ f)(x) = g(f(x)).$$

Definition 9 (injectivity, surjectivity & bijectivity). *A mapping $f\colon M \to N$ is called*

- injective *if for all $x, y \in M$: $f(x) = f(y) \Rightarrow x = y$ (or, equivalently, $x \neq y \Rightarrow f(x) \neq f(y)$).*
- surjective *if $f(M) = N$ (or, equivalently, if for every $y \in N$ there is an $x \in M$ such that $f(x) = y$).*
- bijective *if f is both injective and surjective.*

Definition 10 (inverse mapping). *If $f\colon M \to N$ is bijective, there exists a unique mapping $g\colon N \to M$ such that*

$$(f \circ g)(x) = x \text{ and } (g \circ f)(y) = y$$

for all $x \in M$ and $y \in N$. We write $g = f^{-1}$ and call g the inverse mapping *of f.*

Definition 11 (boundedness). *Let $I \subset \mathbb{R}$ be an interval. A function $f\colon I \to \mathbb{R}$ is called* bounded from above [below] *if there exists a $b \in \mathbb{R}$ such that $f(x) \leq b$ [$f(x) \geq b$] for all $x \in I$.*

Definition 12 (monotonicity). *Let $I \subset \mathbb{R}$ be an interval. A function $f\colon I \to \mathbb{R}$ is called*

- [strictly] monotonically increasing *if for all $x, y \in I$:*

$$x \leq y \Rightarrow f(x) \leq f(y)$$
$$[x < y \Rightarrow f(x) < f(y)]$$

- [strictly] monotonically decreasing *if for all $x, y \in I$:*

$$x \leq y \Rightarrow f(x) \geq f(y)$$
$$[x < y \Rightarrow f(x) > f(y)]$$