Lehrstuhl II für Mathematik http://www.math2.rwth-aachen.de
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6 Determinant

Definition 1 (Determinant)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a matrix with entries $a_{i,j}$.*

- *If $n = 1$, then $A = (a_{1,1})$ and we define*

$$\det(A) = a_{1,1}.$$

- *If $n \geq 2$, we define*

$$\begin{aligned}\det(A) &= a_{1,1}\det(A_1) - a_{2,1}\det(A_2) + - \cdots + (-1)^{n-1}a_{n,1}\det(A_n)\\ &= \sum_{i=1}^{n}(-1)^{i-1}a_{i,1}\det(A_i),\end{aligned}$$

where A_i is obtained from A by deleting the first column and i-th row of A.

$n = 2$:

$$\begin{aligned}\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= a\det(A_1) - c\det(A_2)\\ &= a\det(d) - c\det(b)\\ &= ad - bc\end{aligned}$$

Theorem 2 (Determinant & elementary row transformations)**.** *Let $A, B \in \mathbb{R}^{n\times n}$ such that B is obtained from A by an elementary row transformation.*

(i) If B is obtained from A by swapping two rows (or two columns), then $\det(B) = -\det(A)$.

(ii) If B is obtained from A by multiplication of a row (or column) with a real number c, then $\det(B) = c\det(A)$.

(iii) If B is obtained from A by adding a multiple of a row (or column) to another row (or column), then $\det(B) = \det(A)$.

Definition 3 (Transpose)**.** *The* transpose A^T *of an $(m \times n)$-matrix A with entries $a_{i,j}$ is the $(n \times m)$-matrix B with entries $b_{i,j} = a_{j,i}$.*

Theorem 4 (Invertible matrices & linear systems)**.** *Let $A \in \mathbb{R}^{n\times n}$. Then the following are equivalent.*

(i) *A is invertible.*

(ii) *For every $\mathbf{b} \in \mathbb{R}^n$: $A\mathbf{x} = \mathbf{b}$ has a unique solution.*

(iii) *A has rank n.*

(iv) $\det A \neq 0$.

Theorem 5 (Properties of the determinant)**.** *Let $A \in \mathbb{R}^{n\times n}$ with entries $a_{i,j}$ and let $B, C \in \mathbb{R}^{n\times n}$.*

(i) *If $A = BC$, then* $\det(A) = \det(B)\det(C)$.

(ii) $\det(A^T) = \det(A)$.

(iii) *If*

$$A = \begin{pmatrix} A_1 & B \\ 0_{n-k,k} & A_2 \end{pmatrix}$$

with $A_1 \in \mathbb{R}^{k\times k}$, $A_2 \in \mathbb{R}^{(n-k)\times(n-k)}$ and $B \in \mathbb{R}^{k\times(n-k)}$, then

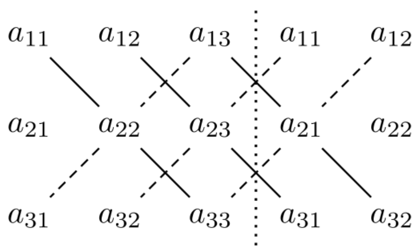
$$\det(A) = \det(A_1)\det(A_2).$$

(iv) *If A is in simple form, then*

$$\det(A) = a_{1,1}a_{2,2} \cdot \ldots \cdot a_{n,n}.$$

(v) *Sarrus' rule is a method and a memorization scheme to compute the determinant of a 3×3 matrix.*

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$



Write out the first 2 columns of the matrix to the right of the 3rd column, so that you have 5 columns in a row. Then add the products of the diagonals going from top to bottom and subtract the products of the diagonals going from bottom to top.

Corollary 6 (Laplace's formula)**.**

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{ij}),$$
$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{ij}),$$

where A_{ij} is obtained from A by deleting the j-th column and i-th row of A.

Theorem 7 (Cramer's rule)**.** *Let $A \in \mathbb{R}^{n\times n}$ invertible with column vectors a_i for $i = 1, \cdots, n$ and $b \in \mathbb{R}^n$. Then the unique solution vector $\mathbf{x} = (x_1, \cdots, x_n)^T$ of the system $A\mathbf{x} = \mathbf{b}$ can be calculated in the following way:*

$$x_i = \frac{1}{\det(A)} \det(a_1, \cdots, a_{i-1}, b, a_{i+1}, \cdots, a_n), \text{ for } i = 1, \cdots, n.$$

Corollary 8. *The Inverse of a 2×2 matrix can be computed as follows:*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$