Lehrstuhl II für Mathematik `http://www.math2.rwth-aachen.de`
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5 Vectors and Spans

Definition 1 (Row vector). *For $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in \mathbb{R}$ an n-*tuple *or* row vector *is defined by $(x_1, x_2, \ldots, x_n)$. Two row vectors $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are equal if and only if $x_i = y_i$ for $i = 1, 2, \ldots, n$.*

Definition 2 ((Column) vector). *For $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in \mathbb{R}$ a* (column) vector *is defined by*

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (x_1, x_2, \ldots, x_n)^T.$$

We denote $(x_1, x_2, \ldots, x_n)^T$ by $\mathbf{x}$ and $\underbrace{(0, 0, \ldots, 0)^T}_{n \text{ times}}$ by $\mathbf{0}$. The last vector is also called the origin.

Definition 3 (Euclidean space). *The n-*dimensional Euclidean space $\mathbb{R}^n$ *is*

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} = \left\{ (x_1, x_2, \ldots, x_n)^T \colon x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}.$$

Definition 4 (Vector addition & scalar multiplication). *Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T, \mathbf{y} = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$. Then*

- $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)^T$ (vector addition*),*
- *for $s \in \mathbb{R}$: $s \cdot \mathbf{x} = (sx_1, sx_2, \ldots, sx_n)^T$ (*scalar multiplication*). The dot $\cdot$ is usually omitted.*

Theorem 5 (vector-space axioms). *The n-dimensional* Euclidean *space $\mathbb{R}^n$ together with vector addition and scalar multiplication fulfills the so-called* vector-space axioms. *Let $x, y, z \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$.*

V1: $x + y = y + x$.

V2: $(x + y) + z = x + (y + z)$.

V3: $\mathbf{0} = x - x = 0 \cdot x.$

V4: $s(x+y) = sx + sy.$

V5: $(s+t) \cdot x = sx + tx.$

V6: $x = 1x = \mathbf{0} + x.$

Definition 6 (scalar product). *The* scalar product *of two column vectors* $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ *and* $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ *is defined as*

$$\mathbf{x} \cdot \mathbf{y} = \sum_{k=1}^{n} x_k y_k = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

The dot $\cdot$ *is usually omitted.*

Definition 7 (Length of a vector). *The* length *of a vector* $\mathbf{x} \in \mathbb{R}^n$ *is*

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Definition 8 (Distance of points). *The* distance *between* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *is*

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|.$$

Theorem 9 (Properties of scalar product & length). *If* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ *and* $s \in \mathbb{R}$*, then*

(i) $(s\mathbf{x})\mathbf{y} = s(\mathbf{x}\mathbf{y})$,

(ii) $\mathbf{x}\mathbf{y} = \mathbf{y}\mathbf{x}$,

(iii) $(\mathbf{x} + \mathbf{y})\mathbf{z} = \mathbf{x}\mathbf{z} + \mathbf{y}\mathbf{z}$,

(iv) $|\mathbf{x}| \geq 0$ *and* $|\mathbf{x}| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$,

(v) $|s\mathbf{x}| = |s||\mathbf{x}|$,

(vi) $|\mathbf{x}\mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$,

(vii) $||\mathbf{x}| - |\mathbf{y}|| \leq |\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$.

Definition 10 (Orthogonality). *Two vectors* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *are* orthogonal *or* perpendicular*, denoted by* $\mathbf{x} \perp \mathbf{y}$*, if* $\mathbf{x}\mathbf{y} = 0$.

Definition 11 (Linear combination). *Let* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m} \in \mathbb{R}^n$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$. *Then*

$$\mathbf{w} = \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \cdots + \lambda_m \mathbf{v_m}$$

is called a linear combination *of* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}$.

Definition 12 (Span). *Let* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m} \in \mathbb{R}^n$. *The* span *of* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}$, *denoted by* $\text{span}(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m})$, *is the set of all linear combinations of* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}$, *i.e.*

$$\text{span}(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}) = \{\mathbf{w} = \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \cdots + \lambda_m \mathbf{v_m} \in \mathbb{R}^n \colon \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}\}.$$

Theorem 13. *Let* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m} \in \mathbb{R}^n$. *Then the set* $U = \text{span}(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m})$ *is a subspace of* $\mathbb{R}^n$, *i.e. it fulfills*

S1: $\mathbf{0} \in U$.

S2: If $x, y \in U$, *then* $x + y \in U$.

S3: If $s \in \mathbb{R}$ *and* $x \in U$, *then* $sx \in U$.

Definition 14 (Linear (in-)dependence). *A set* $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\} \subset \mathbb{R}^n$ *of vectors is called* linearly independent *if*

$$\lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \cdots + \lambda_m \mathbf{v_m} = \mathbf{0} \Longrightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_m = 0.$$

I.e. the only possibility to represent $\mathbf{0} \in \mathbb{R}^n$ *as a linear combination of* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}$ *is choosing all coefficients* $\lambda_1, \lambda_2, \ldots, \lambda_m$ *equal to* 0.

Definition 15 (Basis, Dimension). *A linearly independent set* $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\} \subset \mathbb{R}^n$ *with* $\text{span}(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}) = U$ *resp.* $\mathbb{R}^n$ *is called* basis *of* U *resp.* $\mathbb{R}^n$. *The number* m *is called the* dimension *of* U. *(If* $\text{span}(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}) = \mathbb{R}^n$, *then* $m = n$. *)*

Theorem 16. *Let* $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\} \subset \mathbb{R}^n$ *and* $U \subseteq \mathbb{R}^n$ *a subspace. The following assertions are equivalent:*

- $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ *is a minimal spanning set of* U.
- $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ *is a maximal linearly independent set of vectors of* U.
- *Every* $\mathbf{u} \in U$ *has a unique expression as linear combination of* $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$.
- $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ *is a basis of* U.

Definition 17. *Let a* $(m \times n)$ *matrix* A *be given. Then the set of solutions of the homogenous system* $S(A, \mathbf{0})$ *is called the* kernel *of* A.

Theorem 18 (Structure of the solution set of a homogenous system). *The set of solutions of a homogenous system of linear equations is a span of linearly independent vectors.*

Theorem 19 (Structure of the solution set of an inhomogenous system)**.** *Let* $\mathbf{x}_{sp}$ *be a (special) solution of an inhomogenous system of linear equations and let* S_{hom} *denote the set of all solutions of the corresponding homogenous system. Then the set of solutions of the inhomogenous system is given by*

$$S = \left\{ \mathbf{x} = \mathbf{x}_{sp} + \mathbf{x}_{hom} \colon \mathbf{x}_{hom} \in S_{hom} \right\},$$

i.e. one gets all solutions of the inhomogenous system by adding all solutions of the homogenous system to one solution of the inhomogenous system.