Lehrstuhl II für Mathematik http://www.math2.rwth-aachen.de
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3 Matrices

Definition 1 (Matrix-vector-product). *Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^n$ such that*

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ & \vdots & & \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Then

$$A\mathbf{x} = \begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \end{pmatrix} \in \mathbb{R}^m.$$

Definition 2. *If the matrices A and B have the same size, then their sum is the matrix $A + B$ defined by*

$$(A+B)_{i,j} = (a_{i,j} + b_{i,j}).$$

Definition 3. *A matrix A can be multiplied by a scalar c to obtain the matrix cA, where*

$$(cA)_{i,j} = ca_{i,j}$$

We just multiply each entry of A by c.

Definition 4 (Matrix-matrix-product). *If the number of columns of $A \in \mathbb{R}^{m\times n}$ equals the number of rows of $B \in \mathbb{R}^{n\times k}$, then the product $AB \in \mathbb{R}^{m\times k}$ is defined by*

$$(AB)_{i,j} = \sum_{l=1}^{n} a_{i,l} b_{l,j}.$$

Theorem 5 (Properties of matrix-matrix-product). *Let $A, A' \in \mathbb{R}^{m\times n}$ and $B, B' \in \mathbb{R}^{n\times k}$. Then*

(i) $0_{k,m}A = 0_{k,n}$ *and* $A0_{n,k} = 0_{m,k}$, *where $0_{m,n}$ is the $(m \times n)$-matrix containing only zeroes.*

(ii) $A(B + B') = AB + AB'$ *and* $(A + A')B = AB + A'B$.

(iii) *For* $C \in \mathbb{R}^{k \times \ell}$: $A(BC) = (AB)C$.

Definition 6 (Rank)**.** *The* rank *of a matrix* $A \in \mathbb{R}^{m \times n}$ *is the dimension of the linear space spanned by its rows resp. columns.*

Definition 7 (Identity matrix)**.** *For* $n \in \mathbb{N}$ *the* identity matrix I_n *is defined as*

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathbb{R}^{n \times n},$$

i.e. the entry $a_{i,j}$ *in the i-th row and j-th column is* 1 *if and only if* $i = j$, *and* 0 *otherwise.*

Theorem 8 (Multiplication with the identity matrix)**.** *The identity matrix is the neutral element regarding multiplication:*

(i) $I_n\mathbf{x} = \mathbf{x}$ *for* $\mathbf{x} \in \mathbb{R}^n$;

(ii) $I_m A = A = AI_n$ *for* $A \in \mathbb{R}^{m \times n}$.

Definition 9 (Inverse matrix)**.** *Let* $A \in \mathbb{R}^{n \times n}$. *Then* $B \in \mathbb{R}^{n \times n}$ *is called* inverse *of* A, *denoted by* A^{-1}, *if*

$$AB = I_n = BA.$$

If A *has an inverse, it is called* invertible *or* regular.

Theorem 10 (Invertibility & the product of matrices)**.** *Let* $A, B \in \mathbb{R}^{n \times n}$.

(i) *If* $AB = I_n$ *or* $BA = I_n$, *then* $B = A^{-1}$.

(ii) *If* A *and* B *are invertible, then* AB *is invertible, and* $(AB)^{-1} = B^{-1}A^{-1}$.