Lehrstuhl II für Mathematik http://www.math2.rwth-aachen.de
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1 Numbers

Definition 1. *Important sets of numbers.*

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ *set of natural numbers,*
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$*,*
- $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$ *set of integers,*
- $\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\} \right\}$ *set of rational numbers,*
- $\mathbb{R}$ *set of all decimal expansions, set of real numbers.*

Definition 2 (Prime Number). *A prime number or prime is a natural number greater than one that is divisible by only one and itself.* $\mathbb{P} := \{p \in \mathbb{N} \mid p \text{ prime}\}$

Theorem 3 (Fundamental theorem of arithmetic). *Every positive integer $n > 1$ can be represented in exactly one way as a product of prime powers:*

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

where $p_1 < p_2 < ... < p_k$ are primes and the a_i are positive integers.

Theorem 4 (Euclid). *There are infinitely many primes.*

Definition 5 (axioms of real numbers). *Let $x, y, z \in \mathbb{R}$.*
Axioms of addition

- *neutral element:* $x + 0 = x$*,*
- *associativity:* $(x + y) + z = x + (y + z)$*,*
- *commutativity:* $x + y = y + x$*,*
- *inverse element:* $y + (-y) = 0$*;*

Axioms of multiplication

- *neutral element:* $x \cdot 1 = x$,
- *associativity:* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
- *commutativity:* $x \cdot y = y \cdot x$,
- *inverse element:* $y \cdot \frac{1}{y} = 1$ *for* $y \neq 0$;
- *distribution:* $x \cdot (y + z) = x \cdot y + x \cdot z$.

We write xy for $x \cdot y$, $x - y$ for $x + (-y)$, y^{-1} for $\frac{1}{y}$, $\frac{x}{y}$ for $x \cdot \frac{1}{y}$, $x + x + \cdots + x = nx$, and $x \cdot x \cdot \ldots \cdot x = x^n$. Furthermore, we define $x^0 = 1$ for all $x \in \mathbb{R}$.

Theorem 6 (Computation rules for powers). *If* $n, m \in \mathbb{N}_0$ *and* $x, y \in \mathbb{R}$, *then*

(i) $x^n x^m = x^{n+m}$,

(ii) $x^n y^n = (xy)^n$,

(iii) $(x^n)^m = x^{nm}$.

Theorem 7 (Existence of non-rational numbers). *There exists no rational number* x *with the property that* $x^2 = 2$.

Definition 8 (Ordering Axioms). *For all* $x, y \in \mathbb{R}$ *we have* $x < y$ *or* $x = y$ *or* $y < x$ *and these three possibilities are mutually exclusive. Moreover, for all* $x, y, z \in \mathbb{R}$ *the relation* $<$ *has the following properties:*

a) $x < y$ *and* $y < z$ *implies* $x < z$.

b) $x < y$ *and* $z > 0$ *implies* $x + z < y + z$.

c) $x < y$ *and* $z > 0$ *implies* $x \cdot z < y \cdot z$.

Definition 9 (Minimum/Maximum). *Let* $M \subset \mathbb{R}$ *be a subset of the real numbers. An element* $x \in M$ *is a minimum [maximum] of* M *if* $x \leq y$ *[*$x \geq y$*] for every element* $y \in M$. *We write* $x = \min M$ *[*$x = \max M$*].*

Theorem 10 (Uniqueness of minima/maxima). *If both* x *and* y *are minima/maxima of a set* $M \subset \mathbb{R}$, *then* $x = y$.

Definition 11 (Absolute value)**.** *The* absolute value *or* modulus *of a real number x is defined by*

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}.$$

Theorem 12 (Properties of the absolute value)**.** *Let $x, y \in \mathbb{R}$. Then*

- $|x| \geq 0$ *and* $x = 0$ *precisely if* $x = 0$*,*
- $|xy| = |x||y|$*, in particular* $|x| = |-x|$*,*
- $|x + y| \leq |x| + |y|$ *(*triangle inequality*),*
- $-|x| \leq x \leq |x|$*,*
- *if* $|x| \leq y$*, then* $-y \leq x \leq y$*.*

Definition 13 (Negative powers)**.** *For $n \in \mathbb{N}$ and $x \in \mathbb{R} \setminus \{0\}$ we define*

$$x^{-n} = (x^{-1})^n = \frac{1}{x^n}.$$

Theorem 14 (Existence of roots)**.** *For $n \in \mathbb{N}$ and $x > 0$ there is a unique number $y > 0$ such that $y^n = x$.*

Definition 15 (Roots)**.** *In the situation of Theorem 14, we define*

$$y = x^{\frac{1}{n}} = \sqrt[n]{x}$$

*and call y the n-*th root *of x. For $p, q \in \mathbb{Q}$, $q \neq 0$, we define*

$$x^{\frac{p}{q}} = (x^{\frac{1}{q}})^p.$$

Proposition 16 (Squares are positive)**.** *If $x \in \mathbb{R} \setminus \{0\}$, then $x^2 > 0$.*

Definition 17 (Sums & Products)**.** *For $n, m \in \mathbb{Z}$ with $m \leq n$ and $a_m, a_{m+1}, \ldots, a_n \in \mathbb{R}$ we define*

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n,$$
$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot \ldots \cdot a_n.$$

Complex Numbers

In mathematics, we do a lot of solving of polynomial equation s, which amounts to finding a root of a polynomial. For example, the solutions to the equations $x^2 = 1$ are the same as the solutions of $x^2 - 1 = 0$, that is, they are the roots of the polynomial $x^2 - 1$. These roots are $x = 1$ and $x = -1$. However, some polynomial equations have no real number solutions: for example, the equation

$$x^2 + 1 = 0$$

has no real number solutions, because $x^2 + 1 \geq 1$ if x is a real number. The complex numbers were invented to provide solutions to polynomial equations.

Definition 18. *A complex number is a number z of the form $z = x + iy$, where x and y are real numbers, and i is another number such that $i^2 = -1$. The set $\mathbb{C}$ of* complex numbers *is defined as*

$$\mathbb{C} = \{(x, y) \colon x, y \in \mathbb{R}\}$$

together with an addition *and* multiplication*:*

$$(x, y) + (u, v) = (x + u, y + v),$$
$$(x, y) \cdot (u, v) = (xu - yv, xv + yu).$$

For the imaginary unit $i = (0, 1)$ *we have* $i^2 = (-1, 0)$.

$$z = x + iy,\ x, y \in \mathbb{R}$$

is the standard description of complex numbers $z \in \mathbb{C}$. We call

$$\operatorname{Re} z = x \textit{ and } \operatorname{Im} z = y$$

the real *and* imaginary part *of z. The* complex conjugate *of z is defined by*

$$\bar{z} = x - iy = \operatorname{Re} z - i\operatorname{Im} z.$$

With the above definitions we have

$$z + \bar{z} = 2\operatorname{Re} z,$$
$$z - \bar{z} = 2i\operatorname{Im} z,$$
$$z \cdot \bar{z} = (x + iy)(x - iy) = x^2 + y^2 \in \mathbb{R}.$$

Theorem 19. *All axioms regarding sums and products in $\mathbb{R}$ carry over to $\mathbb{C}$. In particular,*

(i) addition is commutative and associative;

(ii) multiplication is commutative and associative;

(iii) the distributive law relating addition and multiplication holds;

(iv) for every $z \in \mathbb{C}$: $z = z \cdot 1 = z + 0$ and $z \cdot 0 = 0$.

Definition 20. *The distance of a complex number z from the origin is called* modulus, length *or* absolute value *of z and is given by*

$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = \sqrt{z\bar{z}}.$$

Theorem 21. *For $z, w \in \mathbb{C}$:*

a) $|z| = |-z| = |\bar{z}|$;

b) $|z| \geq 0$ *and* $|z| = 0$ *iff* $z = 0$;

c) $|zw| = |z||w|$;

d) $||z| - |w|| \leq |z + w| \leq |z| + |w|$.

Theorem 22. *For $0 \neq z = x + iy$:*

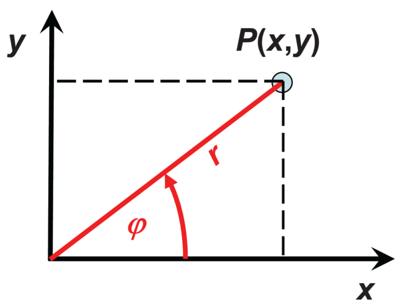
$$(x + iy)\frac{x - iy}{x^2 + y^2} = 1.$$

Hence, the denominator of a quotient $\frac{z}{w}$ can be made a real number by multiplication with $\frac{\bar{w}}{\bar{w}}$.

Theorem 23. *For $z \in \mathbb{C}$ there exist real numbers $r \geq 0$ and $\varphi \in \mathbb{R}$ with*

$$z = r(\cos\varphi + i \sin\varphi).$$

We always have $r = |z|$, and for $z \neq 0$, the number φ is uniquely determined by the condition $-\pi < \varphi \leq \pi$. The pair (r, φ) are the polar coordinates *of z, and φ is called the* argument *of z, denoted by* $\arg z$.



Theorem 24. *For* $\varphi \in \mathbb{R}$*:*

$$e^{i\varphi} = \exp(i\varphi) = \cos\varphi + i\sin\varphi.$$

Theorem 25. *For complex numbers* $z_1 = r_1 e^{i\varphi_1}$ *and* $z_2 = r_2 e^{i\varphi_2}$*:*

$$z_1 \cdot z_2 = r_1 r_2 e^{i(\varphi_1 + \varphi_2)}.$$

Theorem 26 (Formula of de Moivre)**.** *For* $n \in \mathbb{N}$ *and* $w \in \mathbb{C} \setminus \{0\}$ *there exist exactly* n *different solutions of the equation* $z^n = w$ *given by*

$$z_k = \sqrt[n]{|w|} e^{i\frac{\psi + 2k\pi}{n}}, k = 0, 1, \ldots, n-1,$$

where $\psi = \arg w$*.*

Theorem 27 (Fundamental Theorem of Algebra)**.** *Every polynomial with coefficients in* $\mathbb{C}$ *has a root in* $\mathbb{C}$*.*