Lehrstuhl II für Mathematik http://www.math2.rwth-aachen.de
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Analytic geometry

Definition 1 (Scalar product)**.** *The* scalar product *of two column vectors* $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ *and* $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ *is defined as*

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

The dot $\cdot$ *is usually omitted.*

Theorem 2 (Properties of scalar product & length)**.** *If* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ *and* $s \in \mathbb{R}$*, then*

(i) $(s\mathbf{x})\mathbf{y} = s(\mathbf{x}\mathbf{y})$,

(ii) $\mathbf{x}\mathbf{y} = \mathbf{y}\mathbf{x}$,

(iii) $(\mathbf{x} + \mathbf{y})\mathbf{z} = \mathbf{x}\mathbf{z} + \mathbf{y}\mathbf{z}$,

(iv) $|\mathbf{x}| \geq 0$ *and* $|\mathbf{x}| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$,

(v) $|s\mathbf{x}| = |s||\mathbf{x}|$,

(vi) $|\mathbf{x}\mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$,

(vii) $||\mathbf{x}| - |\mathbf{y}|| \leq |\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$.

Definition 3 (Orthogonality)**.** *Two vectors* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *are* orthogonal *or* perpendicular*, denoted by* $\mathbf{x} \perp \mathbf{y}$*, if* $\mathbf{x}\mathbf{y} = 0$.

Definition 4 (Straight line)**.** *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ *or* $\mathbb{R}^3$ *with* $\mathbf{b} \neq 0$*. The* straight line *through* $\mathbf{a}$ *with direction* $\mathbf{b}$ *is the set*

$$L = \{\mathbf{x} = \mathbf{a} + \lambda\mathbf{b} \colon \lambda \in \mathbb{R}\}.$$

This description is called parametric form*, and* $\mathbf{b}$ *is called the* direction vector *of* L.

Theorem 5 (Alternative descriptions of lines)**.** *(i) If* $\mathbf{a} \neq \mathbf{b}$ *are two points in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*, then the unique straight line through both of them is given by*

$$L = \{\mathbf{x} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) \colon \lambda \in \mathbb{R}\}.$$

(ii) If $L \subset \mathbb{R}^2$ *is a straight line, there exists a vector* $\mathbf{n} \in \mathbb{R}^2$ *with* $|\mathbf{n}| = 1$ *and a number* $d \geq 0$ *such that*

$$L = \{\mathbf{x} \in \mathbb{R}^2 \colon \mathbf{n} \cdot \mathbf{x} = d\}.$$

This decription is called HESSE's normal form*, and* $\mathbf{n}$ *is called the* normal vector *of* L.

Definition 6 (Plane)**.** *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^2$ *or* $\mathbb{R}^3$ *such that* $\{\mathbf{b}, \mathbf{c}\}$ *is linearly independent. The* plane *through* $\mathbf{a}$ *spanned by* $\mathbf{b}$ *and* $\mathbf{c}$ *is the set*

$$\begin{aligned} P &= \{\mathbf{x} = \mathbf{a} + \mathbf{y} \colon \mathbf{y} \in \operatorname{span}\{\mathbf{b}, \mathbf{c}\}\} \\ &= \{\mathbf{x} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c} \colon \lambda, \mu \in \mathbb{R}\}. \end{aligned}$$

This description is called parametric form*, and* $\mathbf{b}$ *and* $\mathbf{c}$ *are called the* spanning vectors *of* P.

Theorem 7 (Alternative descriptions of planes). *(i) If* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *are three points in* $\mathbb{R}^3$ *that do not lie on a line (i.e.,* $\{\mathbf{b} - \mathbf{a}, \mathbf{c} - \mathbf{a}\}$ *is linearly independent), then the unique plane through them is given by*

$$P = \{\mathbf{x} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}) \colon \lambda, \mu \in \mathbb{R}\}.$$

(ii) If $P \subset \mathbb{R}^3$ *is a plane, there exists a vector* $\mathbf{n} \in \mathbb{R}^3$ *with* $|\mathbf{n}| = 1$ *and a number* $d \geq 0$ *such that*

$$P = \{\mathbf{x} \in \mathbb{R}^3 \colon \mathbf{n} \cdot \mathbf{x} = d\}.$$

This decription is called HESSE's normal form*, and* $\mathbf{n}$ *is called the* normal vector *of* P.

Theorem 8 (Distance). *Let* M *be a line in* $\mathbb{R}^2$ *or a plane in* $\mathbb{R}^3$ *with Hesse's normal form* $P = \{\mathbf{x} \in \mathbb{R}^3$ *resp.* $\mathbb{R}^2 \colon \mathbf{n} \cdot \mathbf{x} = d\}$ *and let* $\mathbf{x}_0$ *be a point in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*, respectively. The* distance *between* $\mathbf{x}_0$ *and* M *is given by*

$$\operatorname{dist}(\mathbf{x}_0, M) = \min_{\mathbf{y} \in M} |\mathbf{x}_0 - \mathbf{y}| = \mathbf{n}\mathbf{x}_0 - d.$$