Lehrstuhl A für Mathematik
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Aachen, March 14, 2008

Calculus and Linear Algebra

Exam

Exercise 1 (20 points).

Let $\alpha, \beta \in \mathbb{R}$. Consider the following system of linear equations:

$$\begin{pmatrix} 2 & -1 & 3 \\ 1 & -1 & 2 \\ -1 & -1 & -\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ \alpha - \beta - 1 \end{pmatrix}.$$

Depending on α and β, discuss the solvability of the given system and determine the set of solutions.

Exercise 2 (14 points).

Determine all vectors $\mathbf{a} \in \mathbb{R}^3$ with $< (-1,1,0)^T, \mathbf{a} >= 1$, $< \mathbf{a}, (-1,1,-2)^T >= 3$ and length $|\mathbf{a}| = \sqrt{6}$.

Exercise 3 (10 points).

Calculate the limit of the sequence $(a_n)_{n\in\mathbb{N}}$.

a) $a_n = \dfrac{(n^6 - 5n)^2 - n^{12}}{n^7}$,

b) $a_n = \left(\dfrac{(n+1)^2}{n^2-1}\right)^n$.

Exercise 4 (12 points).

Determine whether the series $\sum_{n=1}^{\infty} a_n$ converges or diverges, and justify your answer.

a) $a_n = \dfrac{(n!)^2}{(2n)!}$,

b) $a_n = (-1)^n(\sqrt[n]{n} - 1)$.

Exercise 5 (20 points).

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{x+2}{x^2+5}.$$

a) Determine the monotonicity behaviour and local extrema of f.

b) Determine the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

c) Determine the global extrema of f.

d) Compute the Taylor polynomial T_{2,x_0} of f for $x_0 = 0$.

Exercise 6 (12 points).

Let $T > 1$ and let $f : [1, T] \to \mathbb{R}^3$ be defined by

$$f(x) = \begin{pmatrix} \frac{1}{x} \\ \ln x \\ \frac{-2\sqrt{2}}{\sqrt{x}} \end{pmatrix}.$$

Calculate the length $\ell(\mathcal{C})$ of the curve $\mathcal{C}$ parameterised by f in the interval $[1, T]$.

Exercise 7 (12 points).

Let $D \subseteq \mathbb{R}^2$ be defined by

$$D = \{(x, y)^T \in \mathbb{R}^2 : 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant x^2\}.$$

Calculate the following double integral over the domain D:

$$\iint\limits_D 2xe^y \, d(x, y).$$