Robust Optimization & Network Design
Lecture 6

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Lehrstuhl II für Mathematik
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Robust Maximum Flow Revisited

Maximum Flow in directed graph $G = (V, A)$ with uncertain capacities u_a
What flow value z can be guaranteed without fixing the flow?

1st Stage: Fix z independent of realization $u \in \mathcal{U}$
2nd Stage: Find flow x of value at least z for every $u \in \mathcal{U}$

$$\max z$$

$$s.t. \quad \sum_{a \in \delta^+(v)} x_a(u) - \sum_{a \in \delta^-(v)} x_a(u) = \begin{cases} z & v = s \\ -z & v = t \\ 0 & \text{otherwise} \end{cases} \quad \forall u \in \mathcal{U}, v \in V$$

$$x_a(u) \leq u_a(u) \quad \forall u \in \mathcal{U}, a \in A$$

$$z \geq 0, x_a(u) \geq 0$$

Note: 2nd stage does not mean this is done afterwards; optimization

Theorem 1 (Minoux, 2010)

The robust maximum flow problem is solvable in polynomial time

- *for a fixed number of K scenarios*
- *for $\mathcal{U} = [\bar{u}_1 - \hat{u}_1, \bar{u}_1] \times [\bar{u}_2 - \hat{u}_2, \bar{u}_2] \times \cdots \times [\bar{u}_m - \hat{u}_m, \bar{u}_m]$ (with $A = \{1, \ldots, m\}$), i.e., interval scenarios*

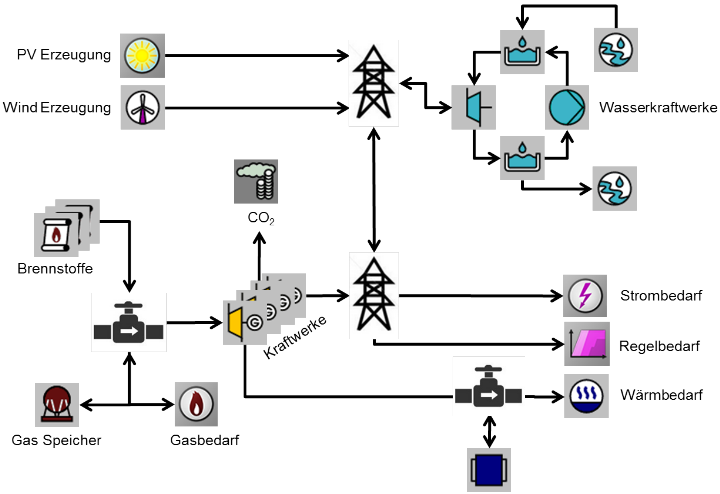
The problem is strongly NP-hard if $\mathcal{U}$ is defined budget uncertainty (Γ-robustness)
The latter problem is polynomial time solvable if G is planar.

The results[1] can be extended to general LPs with uncertain right-hand-side.[2]

[1] M. Minoux. On robust maximum flow with polyhedral uncertainty sets. *Optimization Letters*, 3:367–376, 2009

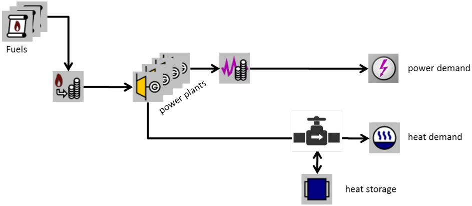
[2] M. Minoux. On 2-stage robust LP with RHS uncertainty: complexity results and applications. *Journal on Global Optimization*, 49:521–537, 2011

Energy System schematically



Source: ProCom

Simultaneous production of *heat* and *power* in exchange for *fuel*



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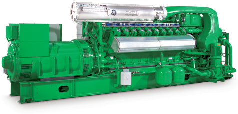
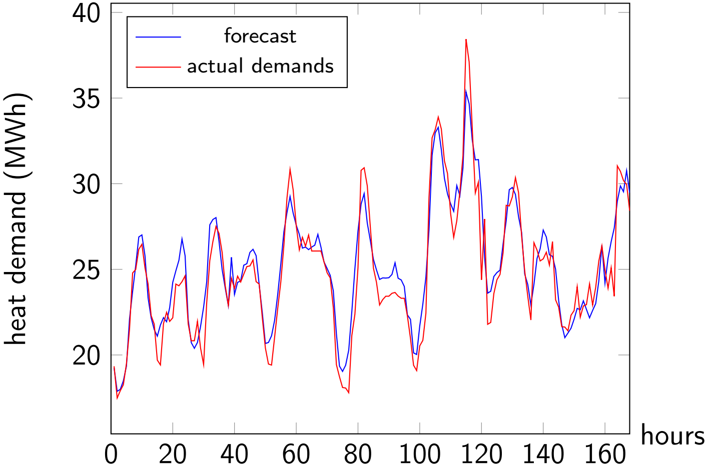


- Fixed ratio ρ between heat and power generation
- Heat can be stored for future use, power cannot be stored
- Heat storage has limited capacity and loss factor

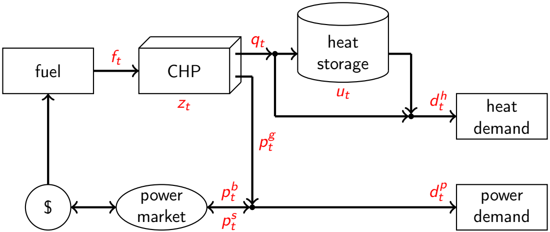
Power has to be bought/sold at day-ahead market!

Heat demands for week 45, 2007



Forecast error of up to 20% (average: 4.1%)

Find solutions that are feasible *with high probability*!



- Time horizon T ($T = 24$)
- Demand for power d_t^p and heat d_t^h
- Fuel f_t to operate CHP costs c_t^f per unit
- Power bought p_t^b/sold p_t^s on day-ahead market at c_t^p per unit
- Generation at time t: p_t^g (power), q_t (heat), z_t (on/off)
- Heat can be stored with loss factor α per time unit
- Storage at end of period t: u_t

The nominal problem

$$\min \quad \sum_{t=1}^{T} \Big(c_t^f f_t(sq_t + hz_t) + c_t^p (p_t^b - p_t^s d_t^p - \rho q_t) \Big) \qquad (1a)$$

$$s.t. \quad \alpha u_{t-1} + q_t = u_t + d_t^h \qquad \forall t \in [T] \qquad (1b)$$

$$p_t^g \rho q_t + p_t^b = d_t^p + p_t^s \qquad \forall t \in [T] \qquad (1c)$$

$$\underline{U} \leq u_t \leq \overline{U} \qquad \forall t \in [T] \qquad (1d)$$

$$\underline{Q} z_t \leq q_t \leq \overline{Q} z_t \qquad \forall t \in [T] \qquad (1e)$$

$$f_t = sq_t + hz_t \qquad \forall t \in [T] \qquad (1f)$$

$$p_t^g = \rho q_t \qquad \forall t \in [T] \qquad (1g)$$

$$q_t, u_t, f_t, p_t^g, p_t^b, p_t^s \geq 0 \qquad \forall t \in [T] \qquad (1h)$$

$$z_t \in \{0, 1\} \qquad \forall t \in [T] \qquad (1i)$$

- Lot-Sizing Problem with Storage Deterioration

LS-DET:

$$\min \quad f(q,z) + \sum_{t=1}^{T} h_t u_t \tag{2a}$$

$$s.t. \quad \alpha u_{t-1} + q_t = u_t + d_t \qquad \forall t \in [T] \tag{2b}$$

$$\underline{U}_t \leq u_t \leq \overline{U}_t \qquad \forall t \in [T] \tag{2c}$$

$$\underline{Q} z_t \leq q_t \leq \overline{Q} z_t \qquad \forall t \in [T] \tag{2d}$$

$$q_t, u_t \geq 0 \qquad \forall t \in [T] \tag{2e}$$

$$z_t \in \{0,1\} \qquad \forall t \in [T] \tag{2f}$$

Lot-Sizing with

- Production limitations
- Storage limitations
- Deterioration of storage
- Concave cost function
- No backlogging

Complexity

- in general: open
- if $\underline{Q} = 0, \overline{Q} = \infty, \alpha = 1$, f linear: LS-DET$\in \mathcal{P}$ (Love, 1973; Atamtürk & Küçükyavuz, 2008)
- if $\underline{U} = 0, \overline{U} = \infty, \alpha = 1$: LS-DET$\in \mathcal{P}$ (Hellion et al., 2012)
- both cases still in $\mathcal{P}$ if $0 < \alpha < 1$ (Schmitz, 2016)

What about uncertain demands?

Uncertainty Set: $\mathcal{U}$ of possible demand realizations

Applying Robust Optimization:
Due to equalities (2b): Impossible to find (q, z, u) such that (2b)–(2f) are satisfied $\forall d \in \mathcal{U}$

Theorem (folklore) Every (implicit) equality in $Ax \leq b$ allows for the elimination of a variable involved in the equality.

$\Rightarrow$ In robust optimization, elimination of variable x implies that this variable is moved 2nd stage, i.e., after the uncertain input is known!

RLS-DET:

$$\min \quad f(q,z) + \eta \tag{3a}$$

$$s.t. \quad \alpha u_{t-1}(d) + q_t = u_t(d) + d_t \qquad \forall t \in [T], d \in \mathcal{U} \tag{3b}$$

$$\underline{U} \leq u_t(d) \leq \overline{U} \qquad \forall t \in [T], d \in \mathcal{U} \tag{3c}$$

$$\eta \geq \sum_{t \in [T]} h^t u_t(d) \qquad \forall d \in \mathcal{U} \tag{3d}$$

$$\underline{Q} z_t \leq q_t \leq \overline{Q} z_t \qquad \forall t \in [T] \tag{3e}$$

$$q_t, u_t(d) \geq 0 \qquad \forall t \in [T] \tag{3f}$$

$$z_t \in \{0,1\} \qquad \forall t \in [T] \tag{3g}$$

$$\eta \geq 0 \tag{3h}$$

- storage $u_t(d)$ per scenario $d \in \mathcal{U}$
- minimization of worst-case storage cost

Solving RLS-DET as LS-DET instance

Theorem 2

For an uncertainty set $\mathcal{U}$ over which a linear function can be optimized in polynomial time, RLS-DET can be polynomially reduced (w.r.t. production plans) to an instance of LS-DET with $d = d'$ and $\overline{U} = \overline{U}'$ thus defined:

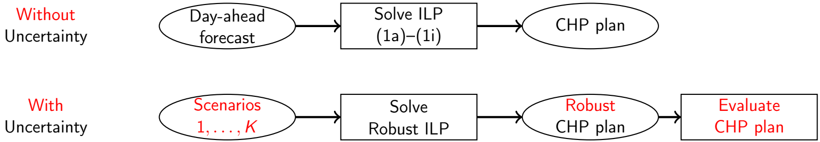
$$d'_t := \max_{d \in \mathcal{U}} \left\{ d_t - \sum_{i=1}^{t-1} \alpha^{t-i} \left(d'_i - d_i \right) \right\} \qquad \forall t \in [T] \tag{4a}$$

$$\overline{U}'_t := \overline{U}_t - \max_{d \in \mathcal{U}} \left\{ \sum_{i=1}^{t} \alpha^{t-i} \left(d'_i - d_i \right) \right\} \qquad \forall t \in [T]. \tag{4b}$$

Corollary 3

Given an uncertainty set $\mathcal{U}$ over which a linear function can be optimized in polynomial time, RLS-DET is in $\mathcal{P}$ (resp., $\mathcal{NP}$-hard) if and only if the corresponding version of LS-DET is in $\mathcal{P}$ (resp., $\mathcal{NP}$-hard).

Robustness models satisfying precondition:



- Operating CHP according to forecast may result in storage violations
- Robust CHP plan will be more costly than forecast-based CHP plan
- Evaluation of CHP reveals real benefits

Computational Setting

Uncertainty Set:

- Discrete Scenarios and Γ-Scenarios based on historical data
- historical data for one year, forecasts available for next 232 days
- $\Rightarrow$ 232 instances with 24h planning horizon
- $\Rightarrow$ 229 instances with 96h planning horizon

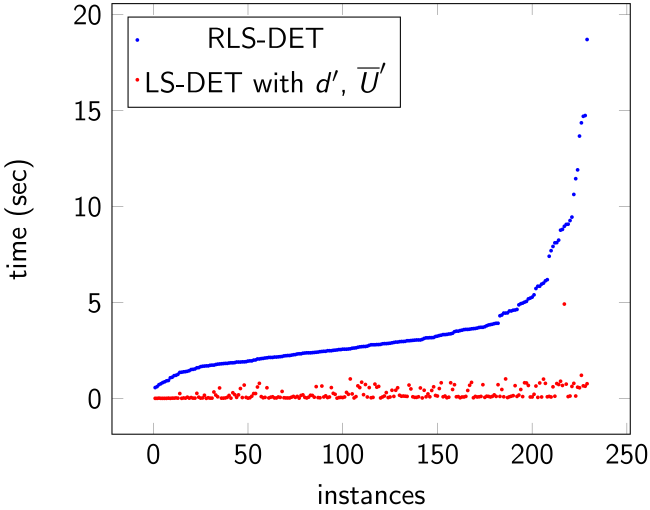
Discrete Scenarios:

- $K = 1, \dots, 70$ considered heat demand scenarios
- $k = 1$: *current forecast*
- $k > 1$: *current forecast* + *forecast error* for similar days in the past

Γ-Scenarios:

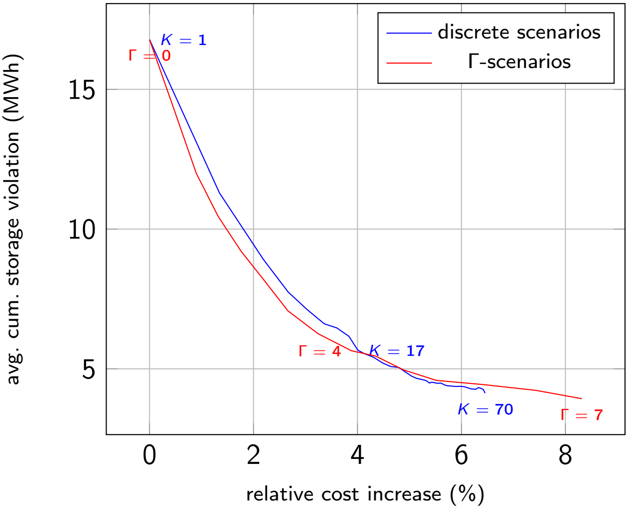
- classify forecast on basis of hour, outside temperature, weekday
- determine deviations by similar historical data

Distribution of running times for $K = 50$:



Speed-up between 1.82 and 85.67 with average 29.00

Evaluation of robust solutions with real heat demand values ($\bar{U} = 120$)



Affine Decision Model

Idea: let production q_t anticipate upon higher/lower demands

- Multi-stage approach: q_t can only anticipate on $d_1, \dots, d_{t-1}$
- Affine decision rule (with nominal demand $\bar{d}_j$):

$$q_t(d) = \sum_{j=1}^{t-1} (d_j - \bar{d}_j) q_t^j + q_t^0$$

- For fixed production plan: $q_t^j := 0$, $j = 1, \dots, t-1$.
- Production of power now depends on considered demand:

$$p_t^g(d) = \rho q_t(d)$$

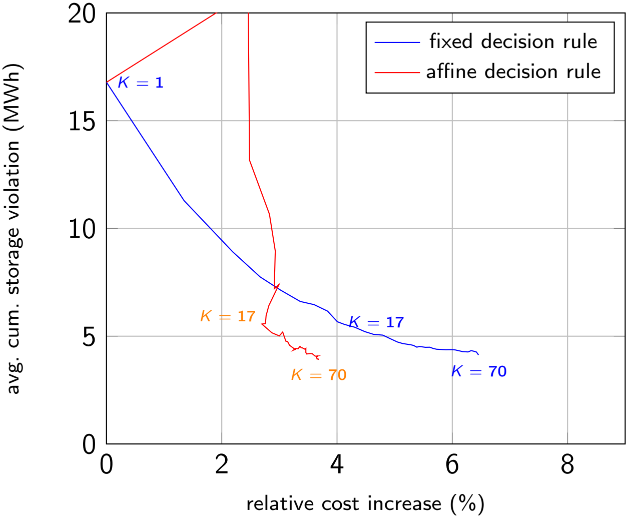
Hence, power balance cannot be guaranteed anymore

$$p_t^g(d) + p_t^b = d_t^p + p_t^s \qquad \forall d \in \mathcal{U}$$

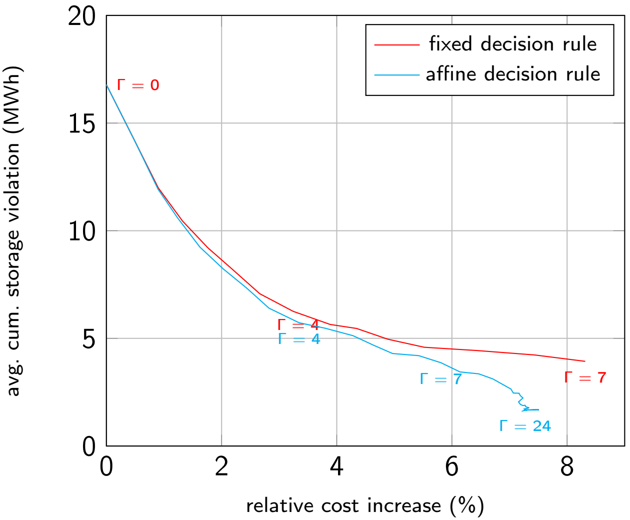
- Difference has to be bought/sold on reserve market:

$$\min \dots + \max_{d \in \mathcal{U}} \left\{ \sum_{t=1}^{T} c_t^R \cdot \left| p_t^g(d) - d_t^p + p_t^b - p_t^s \right| \right\}$$

Discrete Scenarios: Affine Decision Rules



Γ-Scenarios: Affine Decision Rules



Classical Network Design Model

Integer Linear Programming formulation:

$$\min \sum_{a \in A} \sum_{m \in M} \kappa_a^m x_a^m$$

$$\begin{aligned}
s.t. \quad & \sum_{a \in \delta^+(s^k)} f_a^k - \sum_{a \in \delta^-(s^k)} f_a^k && = d^k && \forall k \in K \\
& \sum_{a \in \delta^+(i)} f_a^k - \sum_{a \in \delta^-(i)} f_a^k && = 0 && \forall k \in K, \forall i \in V, i \neq s^k, t^k \\
& \sum_{m \in M} C^m x_a^m - \sum_{k \in K} f_a^k && \geq 0 && \forall a \in A \\
& x_a^m \in \mathbb{Z}_+, f_a^k \geq 0
\end{aligned}$$

Uncertain demand values d^k in right hand side

For robust solution, worst case d^k value have to be taken, say $\bar{d}^k + \hat{d}^k$

Can't we do better? What about dynamic routing?

Dynamic Routing Model

For $d \in \mathcal{U}$, $f_a^k(d)$ denotes the flow on arc a.

$$
\begin{aligned}
\min \; & \sum_{a \in A} \sum_{m \in M} \kappa_a^m x_a^m \\
s.t. \; & \sum_{a \in \delta^+(s^k)} f_a^k(d) - \sum_{a \in \delta^-(s^k)} f_a^k(d) = d^k(d) && \forall d \in \mathcal{U}, k \in K \\
& \sum_{a \in \delta^+(i)} f_a^k(d) - \sum_{a \in \delta^-(i)} f_a^k(d) = 0 && \forall d \in \mathcal{U}, k \in K, i \in V \setminus \{s^k, t \\
& \sum_{m \in M} C^m x_a^m - \sum_{k \in K} f_a^k(d) \geq 0 && \forall d \in \mathcal{U}, a \in A \\
& x_a^m \in \mathbb{Z}_+, f_a^k(d) \geq 0
\end{aligned}
$$

Extremely large model!

Reduction model size by projection on capacity space

For fixed demand, we have:

$$\begin{aligned} P^x &= \operatorname{proj}_x P \\ &= \operatorname{conv}\{x \in \mathbb{Z}_+^{|E|} : \exists\, f \in [0,1]^{|\mathcal{P}|} \text{ such that } (f,x) \in P\} \\ &= \operatorname{conv}\{x \in \mathbb{Z}_+^{|E|} : x \text{ satisfies (5) for all } \ell_M \in \operatorname{Met}(G)\} \end{aligned}$$

where

$$\sum_{e \in E} \ell_M(e) x_e \geq \sum_{k \in K} d^k \ell_M(s^k, t^k), \tag{5}$$

and $\operatorname{Met}(G)$ denotes the cone of metrics in G.

Note, for static (template) routing this result did not hold!

Metric Inequalities for Dynamic Routing[3]

$$\sum_{e \in E} \ell_M(e) x_e \geq \max_{d \in \mathcal{U}} \sum_{k \in K} d^k \ell_M(s^k, t^k), \tag{6}$$

Theorem 4 (Mattia, 2013)

$P^x(\mathcal{U}) = \text{conv}\{x \in \mathbb{Z}_+^{|E|} : x \text{ satisfies (6) for all } \ell_M \in \text{Met}(G)\}$

Theorem 5 (Mattia, 2013)

If $ax \geq b$ is valid for $P^x(\mathcal{U})$, then there exists a metric $\ell_M \in \text{Met}(G)$ such that $\ell_M x \geq b$ is still valid and $\ell_M(e) \leq a_e$ for all $e \in E$.

[3] S. Mattia. The robust network loading problem with dynamic routing. *Computational Optimization and Applications*, 54:619–643, 2013

Dynamic Routing is just one example of two-stage robust optimization:

- At the first stage, some decisions have to be made before the uncertain data is known
- Af the second stage, the uncertain data becomes available and the remaining decisions have to be taken.

Two-stage Robust Optimization vs. Two-stage Stochastic Optimization

$$\begin{aligned} \min c^T x + \max & \begin{bmatrix} \min & q^T z \\ & Bz = h - Tx \\ & z \geq 0 \end{bmatrix} \\ s.t.\ Ax &= b \\ x &\geq 0 \end{aligned}$$

$$\begin{aligned} \min c^T x + \mathbb{E} & \begin{bmatrix} \min & q^T z \\ & Bz = h - Tx \\ & z \geq 0 \end{bmatrix} \\ s.t.\ Ax &= b \\ x &\geq 0 \end{aligned}$$

Static vs. Dynamic Routing

Static Routing:

- Capacities have to be installed in integer amounts
- Routing templates fixes percentual distribution of traffic volume along paths

Dynamic Routing:

- Capacities have to be installed in integer amounts
- Routing can be adapted to actual traffic volumes (realization from uncertainty set)

Do there exist routing models in between static and dynamic routing?

Robust Network Design with Affine Routing:[4]

- Capacities have to be installed in integer amounts
- Routing follows a linear function of all traffic values

$$f_{ij}^{k}(d) := h_{ij}^{k0} + \sum_{\bar{k} \in K} h_{ij}^{k\bar{k}} d^{\bar{k}}$$

where $h_{ij}^{k0}, h_{ij}^{k\bar{k}} \in \mathbb{R}$ for all $ij \in A$, $k, \bar{k} \in K$.

Theorem (Poss & Raack, 2011)

Let $\mathcal{D}$ be an arbitrary demand uncertainty set. Then

$$OPT_{dyn}(\mathcal{D}) \leq OPT_{aff}(\mathcal{D}) \leq OPT_{stat}(\mathcal{D})$$

[4] M. Poss and C. Raack. Affine recourse for the robust network design problem: between static and dynamic routing. *Networks*, 61(2):180–191, 2013

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