Robust Optimization & Network Design
Lecture 4

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Lehrstuhl II für Mathematik
RWTH AACHEN UNIVERSITY

What we did so far?

Chance-constrained Optimization Problems

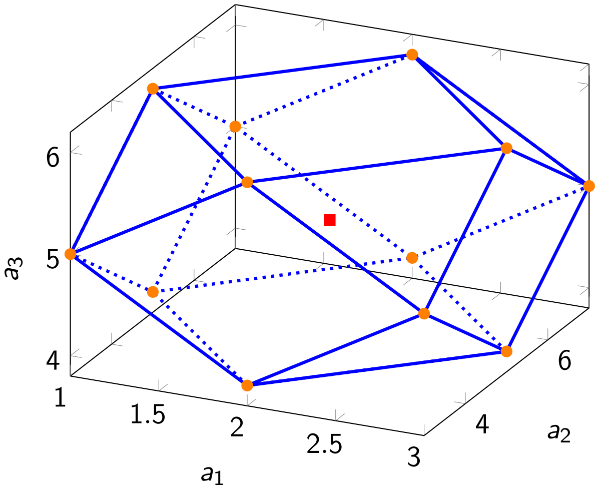
- Chance-constrained Knapsack
- Joint vs. individual chance constraints
- Application from Wireless Backhaul Networks

Uncertain Linear Optimization Problems

- Formalization of uncertainty: uncertainty sets
- Robust Feasible Solutions
- Robust Solution Value
- Robust Counterpart = Single Optimization Problem

Today: Most successful uncertainty set: budget uncertainty

$$\bar{a} = \begin{pmatrix} 2 \\ 5 \\ 5 \end{pmatrix}, \quad \hat{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$



Picture provided by Manuel Kutschka

I.e., exactly two coefficients deviate up or down from a nominal value.

In general, Γ out of n coefficients can deviate

1. Γ-Robust Optimization
2. Example: The Network Loading Problem
3. Network Design under Demand Uncertainty
4. Robust Network Design with Static Routing

Simplifying assumption: b is certain

Uncertainty Set by Bertsimas & Sim (2003,2004): Let $\bar{a}_j \in \mathbb{R}$, $\hat{a}_j \geq 0$ be given, and $\Gamma \in \mathbb{R}_+$ a parameter.

$$\mathcal{U}(\Gamma) = \{a \in \mathbb{R}^n : a_j = \bar{a}_j + \hat{a}_j \zeta_j \quad \forall j = 1, \ldots, n, \quad \zeta \in \mathcal{Z}(\Gamma)\}$$

with

$$\mathcal{Z}(\Gamma) = \left\{ \zeta \in \mathbb{R}^n : |\zeta_j| \leq 1 \quad \forall j = 1, \ldots, n, \quad \sum_{j=1}^{n} |\zeta_j| \leq \Gamma \right\}$$

Stated otherwise:

- nominal values $\bar{a}_j$ and deviations $\hat{a}_j$, $a_j \in [\bar{a}_j - \hat{a}_j, \bar{a}_j + \hat{a}_j]$
- Sum of relative deviations from the nominal values is bounded by Γ

Stated otherwise:

- nominal values $\bar{a}_j$ and deviations $\hat{a}_j$, $a_j \in [\bar{a}_j - \hat{a}_j, \bar{a}_j + \hat{a}_j]$
- Sum of relative deviations from the nominal values is bounded by Γ

At most Γ many entries might deviate from their nominal value

Motivation

If the entries a_j are independently distributed, deviations from the nominal value $\bar{a}_j$ are relatively rare; so only a small number, say Γ, will deviate simultaneously.

Robust Counterpart

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} c_i x_i \\
\text{s.t.} \quad & \sum_{j=1}^{n} \bar{a}_{ij} x_j + \max_{z_i \in \mathcal{Z}_i(\Gamma)} \left(\sum_{j=1}^{n} \hat{a}_{ij} z_{ij} x_j \right) \leq b_i \qquad i = 1, \ldots, m \\
& x \geq 0
\end{aligned}
$$

Observation

Since $\mathcal{Z}_i$ defines a (bounded) polyhedron, only the extreme points have to be treated.

For $\Gamma \in \mathbb{Z}_+$:

$$
\sum_{j=1}^{n} \bar{a}_{ij} x_j + \max_{S \subseteq \{1,\ldots,n\} : |S| \leq \Gamma} \left(\sum_{j \in S} \hat{a}_{ij} x_j \right) \leq b_i \tag{1}
$$

Solving the robust counterpart

$$\sum_{j=1}^{n} \bar{a}_{ij} x_j + \max_{S \subseteq \{1,\dots,n\} : |S| \leq \Gamma} \left(\sum_{j \in S} \hat{a}_{ij} x_j \right) \leq b_i$$

Observations:

- Inequality (1) can be linearized by

$$\sum_{j \notin S} \bar{a}_{ij} x_j + \sum_{j \in S} (\bar{a}_{ij} + \hat{a}_{ij}) x_j \leq b_i \qquad \forall S \subseteq \{1, \dots, n\}, |S| \leq \Gamma \quad (2)$$

- This number of inequalities is exponential if $\Gamma = O(n)$
- Separation can be done in polynomial time

Separation of robust inequalities

Given $x^\star$, find a subset $S \subseteq \{1, \ldots, n\}$ with $|S| \leq \Gamma$ such that

$$\sum_{j \notin S} \bar{a}_{ij} x_j^\star + \sum_{j \in S} (\bar{a}_{ij} + \hat{a}_{ij}) x_j^\star \sum_{j=1}^{n} \bar{a}_{ij} x_j^\star + \sum_{j \in S} \hat{a}_{ij} x_j^\star > b_i - \sum_{j=1}^{n} \bar{a}_{ij} x_j^\star$$

Separation problem:

$$Z_{SEP} = \max \left\{ \sum_{j=1}^{n} \hat{a}_{ij} x_j^\star z_j : \sum_{j=1}^{n} z_j \leq \Gamma, z_j \in \{0,1\} 0 \leq z_j \leq 1 \right\}$$

If $Z_{SEP} > b_i - \sum_{j=1}^{n} \bar{a}_{ij} x_j^\star$, add robust inequality (2) for $S = \{j : z_j = 1\}$.
Optimization = Separation implies polynomial solvability of LP
Alternatively, a compact formulation can be obtained via dualization

Compact formulation

Let $\beta_i(x,\Gamma) = \max_{S \subseteq \{1,\dots,n\}:|S| \leq \Gamma} \left(\sum_{j \in S} \hat{a}_{ij} x_j \right)$, and hence (1) reads

$$\sum_{j=1}^{n} \bar{a}_{ij} x_j + \beta_i(x,\Gamma) \leq b_i$$

Given $x^\star$, $\beta_i(x^\star,\Gamma)$ is the optimization problem

$$\begin{aligned} \beta_i(x^\star,\Gamma) = \max \sum_{j=1}^{n} \hat{a}_{ij} x_j^\star z_j = \quad & \min \Gamma \pi_i + \sum_{j=1}^{n} \rho_{ij} \\ \text{s.t.} \sum_{j=1}^{n} z_j \leq \Gamma \qquad & \text{s.t.} \, \pi_i + \rho_{ij} \geq \hat{a}_{ij} x_j^\star \quad \forall j = 1,\dots,n \\ 0 \leq z_j \leq 1 \qquad & \pi_i, \rho_{ij} \geq 0 \end{aligned}$$

Thus, (1) now reads

$$\sum_{j=1}^{n} \bar{a}_{ij} x_j + \min \left\{ \Gamma \pi_i + \sum_{j=1}^{n} \rho_{ij} : \pi_i + \rho_{ij} \geq \hat{a}_{ij} x_j \quad \forall j, \pi_i \geq 0, \rho_{ij} \geq 0 \right\} \leq b_i$$

or equivalently

$$\begin{aligned} & \sum_{j=1}^{n} \bar{a}_{ij} x_j + \Gamma \pi_i + \sum_{j=1}^{n} \rho_{ij} \leq b_i \\ & \pi_i + \rho_{ij} \geq \hat{a}_{ij} x_j \qquad \forall j = 1, \ldots, n \\ & \pi_i \geq 0, \rho_{ij} \geq 0 \end{aligned}$$

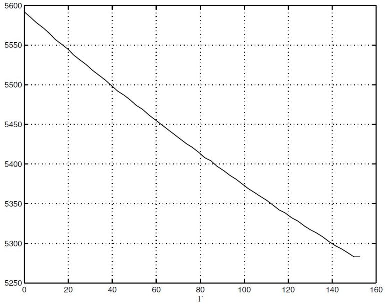
Conclusion: Every uncertain row (including the objective) can be reformulated this way.

Price of Robustness (Bertsimas & Sim)

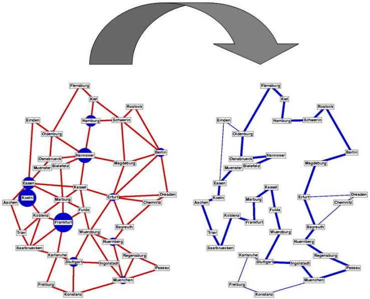
With increasing Γ the optimal solution value degrades: the price of robustness.

Knapsack with uncertain objective:

Optimal value of the robust knapsack formulation as a function of Γ.



Remark: The value $\Gamma = 0$ is far too optimistic as it is likely to realize worse than calculated. Better evaluate the solution regarding "realized robustness".



given a potential network topology, demand forecast, link modules with capacities/costs

find a hardware configuration, and a routing,

such that demands are satisfied and total installation cost is minimised.

discrete decisions:

Variations / Extensions:

- Single path routing
- Integer routing
- Survivability requirements
- Node hardware (switching capacity)
- Wavelength assignment
- Multi-layer scenarios

A First Network Design Model

Notation

Network topology:

- directed graph $G = (V, A)$; set of nodes V, set of arcs A

Installable arc modules:

- Set M; provide arc capacity C^m at cost κ_a^m for all $m \in M$ and $a \in A$

Traffic matrix:

- $D = (d_{vw})$ with $v, w \in V$ and $v \neq w$

Decision variables

Capacity installation:

- $x_a^m \in \mathbb{Z}_+$ number of times module $m \in M$ is installed at arc $a \in A$

Flow variables:

- $f_a^{v,w} \in \mathbb{R}_+$ flow for demand between v and w along arc $a \in A$

Integer Linear Programming formulation:

$$\min \sum_{a\in A}\sum_{m\in M} \kappa_a^m x_a^m$$

$$s.t. \sum_{a\in\delta^+(v)} f_a^{vw} - \sum_{a\in\delta^-(v)} f_a^{vw} = d_{vw} \quad \forall v,w\in V$$

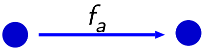
$$\sum_{a\in\delta^+(i)} f_a^{vw} - \sum_{a\in\delta^-(i)} f_a^{vw} = 0 \quad \forall v,w\in V,\ \forall i\in V, i\neq v,w$$

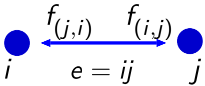
$$\sum_{m\in M} C^m x_a^m - \sum_{v,w\in V} f_a^{vw} \geq 0 \quad \forall a\in A$$

$$x_a^m \in \mathbb{Z}_+, f_a^{vw} \geq 0$$

\# Variables: $O(|A| * |M| + |A| * |V|^2)$
\# Constraints: $O(|V|^3 + |A|)$

Network Design Variants





Directed

$f_a \leq \text{capacity}(a)$

Bidirected

$f_{(i,j)} \leq \text{capacity}(e)$

$f_{(j,i)} \leq \text{capacity}(e)$

Undirected

$f_{(i,j)} + f_{(j,i)} \leq \text{capacity}(e)$

Alternative models

- Bidirected case:

$$\max\left(\sum_{k\in K} f^k_{(i,j)}, \sum_{k\in K} f^k_{(j,i)}\right) \leq \sum_{m\in M} C^m x^m_e \qquad \forall e \in E$$

- Undirected case:

$$\sum_{k\in K}\left(f^k_{(i,j)} + f^k_{(j,i)}\right) \leq \sum_{m\in M} C^m x^m_e \qquad \forall e \in E$$

Modelling Single Path Routing

Routing between v and w can use a single path only:

$$f_a^k \in \{0, d^k\}$$

$\Rightarrow$ cannot be guaranteed by branch-and-bound

Variable transformation:

Let $f_a^k \in \{0,1\}$ denoting usage of link a or not.
Flow of value 1 to be sent from source s^k to target t^k (only one!)

New flow conservation:

$$\sum_{a \in \delta^+(v)} f_a^k - \sum_{a \in \delta^-(v)} f_a^k = \begin{cases} 1 & v = s^k \\ -1 & v = t^k \\ 0 & \text{otherwise} \end{cases} \qquad \forall k \in K, \ \forall v \in V$$

New capacity constraint:

$$\sum_{m \in M} C^m x_a^m - \sum_{k \in K} d^k f_a^k \geq 0 \qquad \forall a \in A$$

Time for Robust Network Design

Question: How can we make a Network Design robust?

Well, that depends on the situation.

Integer Linear Programming formulation:

$$
\begin{aligned}
\min \ & \sum_{a \in A} \sum_{m \in M} \kappa_a^m x_a^m \\
s.t. \ & \sum_{a \in \delta^+(s^k)} f_a^k - \sum_{a \in \delta^-(s^k)} f_a^k && = d^k && \forall k \in K \\
& \sum_{a \in \delta^+(i)} f_a^k - \sum_{a \in \delta^-(i)} f_a^k && = 0 && \forall k \in K, \ \forall i \in V, i \neq s^k, t^k \\
& \sum_{m \in M} C^m x_a^m - \sum_{k \in K} f_a^k && \geq 0 && \forall a \in A \\
& x_a^m \in \mathbb{Z}_+, f_a^k \geq 0
\end{aligned}
$$

Uncertain demand values d^k in right hand side

For robust solution, worst case d^k value have to be taken, say $\bar{d}^k + \hat{d}^k$

Can't we do better? Yes, we can!

Variable transformation:

Let $f_a^k \in \{0,1\}[0,1]$ denoting usage of link a or not.

Flow of value 1 to be sent from s^k to t^k

New flow conservation:

$$\sum_{a \in \delta^+(v)} f_a^k - \sum_{a \in \delta^-(v)} f_a^k = \begin{cases} 1 & v = s^k \\ -1 & v = t^k \\ 0 & \text{otherwise} \end{cases} \qquad \forall k \in K, \ \forall v \in V$$

New capacity constraint:

$$\sum_{m \in M} C^m x_a^m - \sum_{k \in K} d^k f_a^k \geq 0 \qquad \forall a \in A$$

New interpreation: f_a^k denotes percentage of traffic across arc a.

Equivalent Network Design Model

$$
\begin{aligned}
\min \quad & \sum_{a \in A} \sum_{m \in M} \kappa_a^m x_a^m \\
s.t. \quad & \sum_{a \in \delta^+(s^k)} f_a^k - \sum_{a \in \delta^-(s^k)} f_a^k = 1 && \forall k \in K \\
& \sum_{a \in \delta^+(i)} f_a^k - \sum_{a \in \delta^-(i)} f_a^k = 0 && \forall k \in K,\ \forall i \in V, i \neq s^k, t^k \\
& \sum_{m \in M} C^m x_a^m - \sum_{k \in K} d^k f_a^k \geq 0 && \forall a \in A \ \forall d \in \mathcal{U} \\
& x_a^m \in \mathbb{Z}_+, f_a^k \geq 0
\end{aligned}
$$

⇒ Capacity constraints contain all demands simultanously

⇒ Robustness implies worst case of every capacity constraint

⇒ Allows for robustness concepts like Bertimas and Sim!

Static vs. Dynamic Routing

Static Routing

Routing is specified by a template. Traffic is split proportionally to the actual volume.

Real static Routing

Routing along an arc up to a certain amount of traffic. If the traffic volume exceeds the amounts on the outgoing arcs, traffic is dropped.

Dynamic Routing

Routing depends on the actual traffic volumes (all together).
I.e., if $\mathcal{D} \subset \mathbb{R}^{n(n-1)}$ is the set of admissable traffic values (uncertainty set), there exists a routing $f(d)$ for every $d \in \mathcal{D}$.

In this lecture: static routing template

- Let d^k be a random variable with nominal $\bar{d}^k$ and deviation $\hat{d}^k$; $d^k \in [0, \bar{d}^k + \hat{d}^k]$
- Goal: Network satisfying Γ deviated demands simultaneously
- Only capacity constraints change to:

$$\sum_{k \in K} \bar{d}^k f_e^k + \max_{Q \subseteq K, |Q| \leq \Gamma} \left\{ \sum_{k \in Q} \hat{d}^k f_e^k \right\} \leq Cx_e,$$

  All other coefficients are deterministic.
- Linearization by an exponential number of constraints.

Given f, subset $Q \subseteq K$ maximizing the traffic can be computed by linear programming:

$$\max_{Q \subseteq K, |Q| \leq \Gamma} \left\{ \sum_{k \in Q} \hat{d}^k f_e^k \right\} = \max \sum_{k \in K} \hat{d}^k f_e^k z_e^k$$

$$\text{s.t.} \quad \sum_{k \in K} z_e^k \leq \Gamma$$

$$0 \leq z_e^k \leq 1, \quad k \in K$$

where the primal variables $z_e^k = 1$ if $k \in Q$ and 0 otherwise.

Fixing the flow vector f, the subset $Q \subseteq K$ that maximizes the traffic on $e \in E$ can be computed by linear programming:

$$
\begin{aligned}
\max \sum_{k \in K} \hat{d}^k f_e^k z_e^k \quad & = \min \sum_{k \in K} \rho_e^k + \pi_e \Gamma \\
\text{s.t.} \quad \sum_{k \in K} z_e^k \leq \Gamma \quad & \quad \text{s.t.} \quad \rho_e^k + \pi_e \geq \hat{d}^k f_e^k, \quad k \in K \\
0 \leq z_e^k \leq 1, \quad k \in K \quad & \quad \rho_e^k, \pi_e \geq 0, \quad k \in K,
\end{aligned}
$$

where the dual variable $\pi_e \sim$ GUB constraint and $\rho_e^k \sim z_e^k \leq 1$.

Compact Reformulaton:

$$\min \sum_{e \in E} \kappa_e x_e \tag{4}$$

$$s.t. \sum_{ij \in \delta(i)} (f^k_{(i,j)} - f^k_{(j,i)}) = \begin{cases} 1 & i = s(k) \\ -1 & i = t(k) \\ 0 & else \end{cases}, \qquad \forall i \in V,\ k \in K \tag{5}$$

$$\sum_{k \in K} \bar{d}^k f^k_e + \Gamma \pi_e + \sum_{k \in K} \rho^k_e \leq Cx_e, \qquad \forall e \in E \tag{6}$$

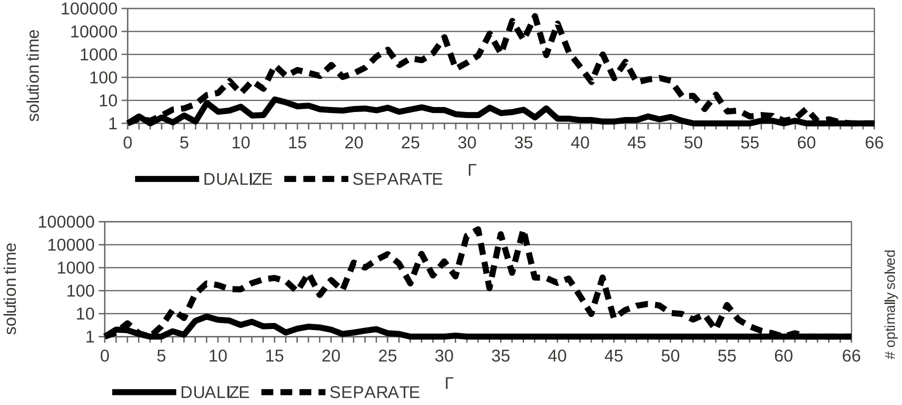
$$\hat{d}^k f^k_e \leq \rho^k_e + \pi_e, \qquad \forall e \in E, k \in K \tag{7}$$

$$f, x, \rho, \pi \geq 0 \tag{8}$$

$$x \in \mathbb{Z}^{|E|} \tag{9}$$

- Compared to the deterministic model, we have $|E| + |E||K|$ additional variables and $|E||K|$ additional constraints.

DUALIZE = compact reformulation
SEPARATE = separation of robust inequalities



⇒ DUALIZE outperforms SEPARATE

Nevertheless, CPU time increases ⇒ new valid inequalities might help!

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