Discrete Optimization under Uncertainty
Lecture 2

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PhD course, Uppsala Universitet – February 26 - March 2, 2018

Lehrstuhl II für Mathematik
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Chance-constrained Optimization Problem (COP)

Find among all solutions that satisfy all constraints with high probability a solution with optimal objective value.

How to solve a COP?

- Stochastic Optimization
  - Modelling with random variables
  - Quite challenging to solve resulting problems
  - Probability distribution have to be determined
- Robust Optimization
  - Uncertainty comes from a known set, the *uncertainty set*
  - No information on probability distribution needed
  - Seeks for solution with best worst-case objective guarantee

Chance-Constrained Linear Programming with joint constraints

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \mathcal{P}\left(Ax \leq b\right) \geq 1 - \epsilon \\ & x \geq 0 \end{aligned}$$

with Entries of A, b and/or c are not constant but random variables

Chance-Constrained Linear Programming with individual constraints

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \mathcal{P}\left(A_i x \leq b_i\right) \geq 1 - \epsilon_i \qquad \forall i = 1, \dots, m \\ & x \geq 0 \end{aligned}$$

with Entries of A, b and/or c are not constant but random variables

Example

Chance-Constrained Knapsack:

Knapsack with n Items, profits c_i, uncertain weights a_i, and capacity b

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} c_i x_i \\ \text{s.t.} \quad & \mathcal{P}\left(\sum_{i=1}^{n} a_i x_i \leq b\right) \geq 1 - \epsilon \\ & x \in \{0,1\}^n \end{aligned}$$

How to solve this problem?

Theorem 1 (Goyal & Ravi, 2009)

If $a_i \in \mathcal{N}(m_i, \sigma_i)$ (independent), the chance-constrained knapsack problem

- *can be formulated as integer second-order-cone program, and*
- *there exists a FPTAS (fully polynomial time approximation scheme).*

Example (cont.)

Assumption: Weights are independent and $a_i \in \mathcal{N}(m_i, \sigma_i)$.

$$\mathcal{P}\left(\sum_{i=1}^{n} a_i x_i \leq b\right) = \mathcal{P}\left(\frac{\sum_{i=1}^{n}(a_i x_i - m_i x_i)}{\sqrt{\sum_{i=1}^{n}\sigma_i^2 x_i^2}} \leq \frac{b - \sum_{i=1}^{n} m_i x_i}{\sqrt{\sum_{i=1}^{n}\sigma_i^2 x_i^2}}\right)$$

$$= \mathcal{P}\left(Z \leq \frac{b - \sum_{i=1}^{n} m_i x_i}{\sqrt{\sum_{i=1}^{n}\sigma_i^2 x_i^2}}\right) \geq 1 - \epsilon$$

with $Z = \frac{\sum_{i=1}^{n}(a_i x_i - m_i x_i)}{\sqrt{\sum_{i=1}^{n}\sigma_i^2 x_i^2}}$

Let $\Phi(.)$ be the cumulative distribution function of the standard normal distribution. Then,

$$\frac{b - \sum_{i=1}^{n} m_i x_i}{\sqrt{\sum_{i=1}^{n}\sigma_i^2 x_i^2}} \geq \Phi^{-1}(1 - \epsilon)$$

Example (cont.)

$$\frac{b - \sum_{i=1}^{n} m_i x_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i^2}} \geq \Phi^{-1}(1-\epsilon)$$

If $1 - \epsilon > 0.5$, $\Phi^{-1}(1-\epsilon) > 0$ and the chance constrained knapsack can be reformulated as

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} c_i x_i \\ \text{s.t.} \quad & \Phi^{-1}(1-\epsilon)\sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i^2} + \sum_{i=1}^{n} m_i x_i \leq b \\ & x \in \{0,1\}^n \end{aligned}$$

After relaxing the integrality of x, a second order cone problem remains, which can be solved in polynomial time.

Example

Chance-Constrained Knapsack:

Knapsack with n Items, profits c_i, uncertain weights a_i, and capacity b

k weight vectors: a^j with probabilities $p^j \in [0,1]$, $j = 1, \ldots, k$

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} c_i x_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} a_i^j x_i \leq b + M(1 - y^j) \qquad j = 1, \ldots, k \\
& \sum_{j=1}^{k} p^j y^j \geq 1 - \epsilon \\
& x \in \{0,1\}^n, y \in \{0,1\}^k
\end{aligned}
$$

How to solve this problem?

Introduce new variables y^j denoting whether solution is feasible in scenario j

Joint vs. Individual Chance-Constraints

Chance-Constrained Linear Programming with joint constraints

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \mathcal{P}\left(Ax \leq b\right) \geq 1 - \epsilon \\ & x \geq 0 \end{aligned}$$

Chance-Constrained Linear Programming with individual constraints

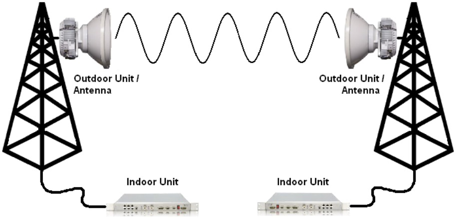
$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \mathcal{P}\left(A_i x \leq b_i\right) \geq 1 - \epsilon_i \qquad \forall i = 1, \ldots, m \\ & x \geq 0 \end{aligned}$$

Question: Which approach is better?

Fixed broadband wireless communications

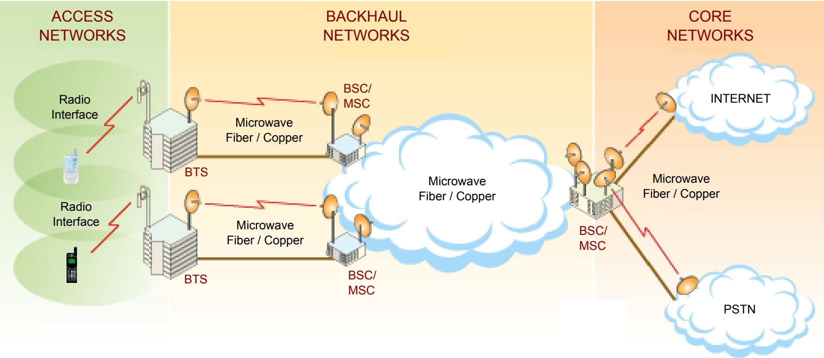
Fixed broadband wireless communications refer to terrestrial point-to-point digital microwave radio transmission.

- Highly directional antennae
- Clear line-of-sight (LOS)
- Licensed frequency bands
- Distance up to 50 km
- Capacity up to 500 Mbps



Wireless backhaul networks

The backhaul is the portion of the network infrastructure that provides interconnectivity between the access and core networks.

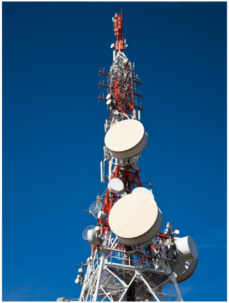


Technologies: copper, fiber, and microwave.

Microwave advantages

- Rapid deployment
- Economical cost
- Capacity improvements
- Disaster resiliency

Over 50% of the world's mobile base stations are connected using microwave technologies!
(Little, 2009)



S. Little. Is microwave backhaul up to the 4G task? IEEE Microwave Magazine, 2009.

Microwave links are time-varying and present a dynamic behavior.

Environment conditions

⇓

Channel fluctuation

⇓

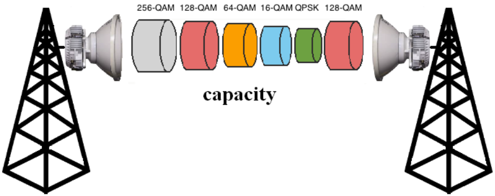
Capacity variation

⇓

Adaptive modulation

⇓

Performance criteria

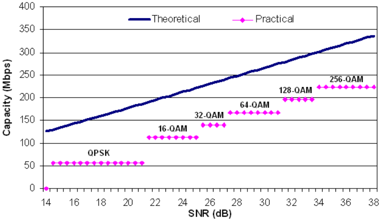


Theoretical capacity: (Shannon, 1948)

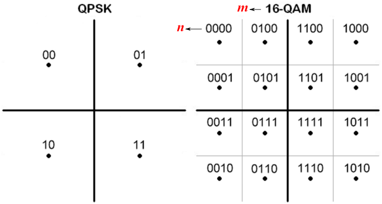
$$C[bps] = B[Hz] \cdot \log_2 \left(1 + \frac{S[W]}{N[W]}\right)$$

Practical bitrate:

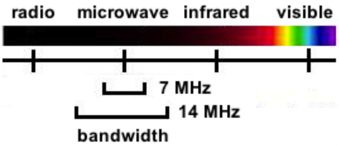
$$C[bps] = n \cdot B[Hz], n = \log_2 m$$



Modulation scheme	Bandwidth efficiency	SNR requirement	Capacity for 7 MHz	Capacity for 28 MHz
QPSK	2 bps/Hz	14.21 dB	14 Mbps	56 Mbps
16-QAM	4 bps/Hz	21.02 dB	28 Mbps	112 Mbps
32-QAM	5 bps/Hz	25.24 dB	35 Mbps	140 Mbps
64-QAM	6 bps/Hz	27.45 dB	42 Mbps	168 Mbps
128-QAM	7 bps/Hz	31.10 dB	49 Mbps	196 Mbps
256-QAM	8 bps/Hz	33.78 dB	56 Mbps	224 Mbps



An assignment (license) is the authorization given by an administration for a radio station to use a radio frequency.

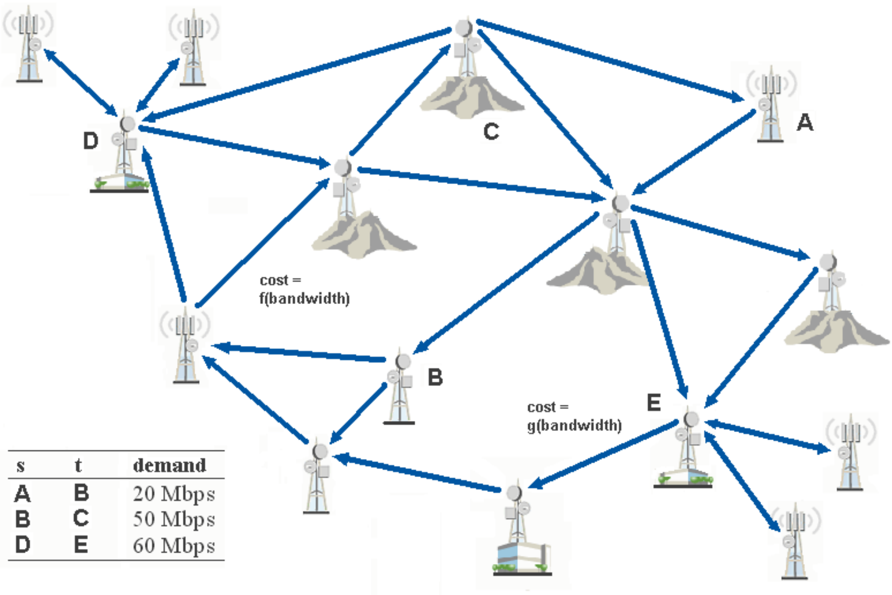


- Auction
- Annual fees

Capacity of a link depends on bandwidth reserved and modulation scheme in use

How to build economical and reliable fixed wireless networks?

Bandwidth assignment problem



s	t	demand
A	B	20 Mbps
B	C	50 Mbps
D	E	60 Mbps

Individual Chance-constrained Model

- Bandwidth options W_{uv} with cost c_{uv}^w, uncertain capacity B_{uv}^w; demands d^k
- Variables: $y_{uv}^w \in \{0,1\}$ (bandwidth decision) and $f_{uv}^k \in [0,1]$ (flow percentage)

$$\min \sum_{uv \in E} \sum_{w \in W_{uv}} c_{uv}^w y_{uv}^w \tag{1}$$

$$s.t. \sum_{u \in \delta^-(v)} f_{uv}^k - \sum_{u \in \delta^+(v)} f_{vu}^k = \begin{cases} -1, & \text{if } v = s^k, \\ 1, & \text{if } v = t^k, \forall v \in V, k \in K \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

$$\mathcal{P}\left(\sum_{k \in K} d^k f_{uv}^k \leq \sum_{w \in W_{uv}} B_{uv}^w y_{uv}^w \right) \geq 1 - \epsilon \qquad \forall uv \in E \tag{3}$$

$$\sum_{w \in W_{uv}} y_{uv}^w = 1 \qquad \forall uv \in E \tag{4}$$

$$f_{uv}^k \in [0,1], y_{uv}^w \in \{0,1\} \tag{5}$$

If ϵ is small, only the lowest modulation scheme at every link guarantees satisfaction of chance constraints (i.e., $B_{uv}^w = b^w * 2bps/Hz$)

Joint Chance-constrained Model

Alternative: Joint Chance-constrained Model

- random variable η_{uv}^{w} denoting spectrum efficiency with (known) discrete probability
- b_{uv}^{w} bandwidth of choice $w \in W_{uv}$

Replace constraints (3) by joint chance constraints

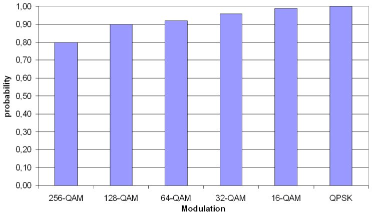
$$\mathcal{P}\left(\sum_{k\in K} d^k f_{uv}^k \leq \sum_{w\in W_{uv}} \eta_{uv}^{w} b_{uv}^{w} y_{uv}^{w} \quad \forall uv \in E\right) \geq 1-\epsilon \tag{6}$$

Chance-constrainted integer linear program! How to obtain a linear model?

Assumption: Independent probabilities

Deterministic MINLP model

- ρ_{uv}^{wm} = probability that link uv can be run with bandwidth w and modulation scheme m or better
- $y_{uv}^{wm} = 1$ iff link uv should run with bandwidth w and modulation scheme m if possible



$$\sum_{k \in K} d^k f_{uv}^k \leq \sum_{w \in W_{uv}} \sum_{m \in M_{uv}^w} b_{uv}^{wm} y_{uv}^{wm} \qquad \forall uv \in E \tag{7}$$

$$\prod_{uv \in E} \left(\sum_{w \in W_{uv}} \sum_{m \in M_{uv}^w} \rho_{uv}^{wm} y_{uv}^{wm} \right) \geq 1 - \epsilon \tag{8}$$

This is a mixed integer non-linear model!

$$\prod_{uv\in E}\left(\sum_{w\in W_{uv}}\sum_{m\in M_{uv}^{w}}\rho_{uv}^{wm}y_{uv}^{wm}\right)\geq 1-\epsilon$$

$$\Leftrightarrow\quad \log\left(\prod_{uv\in E}\sum_{w\in W_{uv}}\sum_{m\in M_{uv}^{w}}\rho_{uv}^{wm}y_{uv}^{wm}\right)\geq \log(1-\epsilon)$$

$$\Leftrightarrow\quad \sum_{uv\in E}\log\left(\sum_{w\in W_{uv}}\sum_{m\in M_{uv}^{w}}\rho_{uv}^{wm}y_{uv}^{wm}\right)\geq \log(1-\epsilon)$$

$$\Leftrightarrow\quad \sum_{uv\in E}\sum_{w\in W_{uv}}\sum_{m\in M_{uv}^{w}}\log(\rho_{uv}^{wm})y_{uv}^{wm}\geq \log(1-\epsilon)$$

Deterministic MILP model

$$\min \sum_{uv \in E} \sum_{w \in W_{uv}} \sum_{m \in M_{uv}^w} c_{uv}^w y_{uv}^{wm} \tag{9}$$

$$s.t. \sum_{u \in \delta^-(v)} f_{uv}^k - \sum_{u \in \delta^+(v)} f_{vu}^k = \begin{cases} -1, & \text{if } v = s^k, \\ 1, & \text{if } v = t^k, \forall v \in V, k \in K \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

$$\sum_{k \in K} d^k f_{uv}^k \leq \sum_{w \in W_{uv}} \sum_{m \in M_{uv}^w} b_{uv}^{wm} y_{uv}^{wm} \qquad \forall uv \in E \tag{11}$$

$$\sum_{uv \in E} \sum_{w \in W_{uv}} \sum_{m \in M_{uv}^w} \log(\rho_{uv}^{wm}) y_{uv}^{wm} \geq \log(1 - \epsilon) \tag{12}$$

$$\sum_{w \in W_{uv}} \sum_{m \in M_{uv}^w} y_{uv}^{wm} = 1 \qquad \forall uv \in E \tag{13}$$

$$f_{uv}^k \in [0,1], y_{uv}^{wm} \in \{0,1\} \tag{14}$$

Kind of Capacitated network design problem

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