Discrete Optimization under Uncertainty
Lecture 1

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Lehrstuhl II für Mathematik | RWTH AACHEN UNIVERSITY

About this course

- Lecture 1: Discrete Optimization w/o Uncertainties
- Lecture 2: Chance-Constrained Optimization
- Lecture 3: Uncertain Linear Optimization & Uncertainty Sets
- Lecture 4: Γ-Robustness
- Lecture 5: More on Γ-Robustness
- Lecture 6: Two-Stage Robust Optimization
- Lecture 7: Recoverable Robust Optimization and more

Lectures:

Monday Feb 26, 2018: 10:15–11:45, ITC 1245

Tuesday Feb 27, 2018: 9:15–10:45 & 11:15–12:45, ITC 2345

Wednesday Feb 28, 2018: 9:15–10:45 & 11:15–12:45, ITC 2345

Thursday Mar 1, 2018: 9:15–10:45 & 11:15–12:45, ITC 1213

Examination: Development of a research proposal, identifying research topics/problems of relevance and presenting the application of the knowledge gained from the course and own reading.

Extra: Seminar centering "Solving Mixed-Integer Non-Linear Programs by Adaptive Discretisation: Two Case Studies", Tuesday Feb 27, 2018, 15:15–16:15, ITC 1245

1 Motivation

2 Discrete Optimization Basics

2.1 Polyhedral Combinatorics

2.2 Example

2.3 Chvátal-Gomory Cuts & Mixed Integer Rounding

2.4 ILP Software

Most important disadvantage of optimization:

"*An optimal solution is as good as the input data is*"

Example: Packing items in bins

Uncertainty in Optimization Problems

Optimization problems often contain uncertain parameters, due to

- measurement/rounding errors
  e.g., temperature, current inventory
- estimation errors
  e.g., demand, cost or prices
- implementation errors
  e.g., length, depth, width, voltage

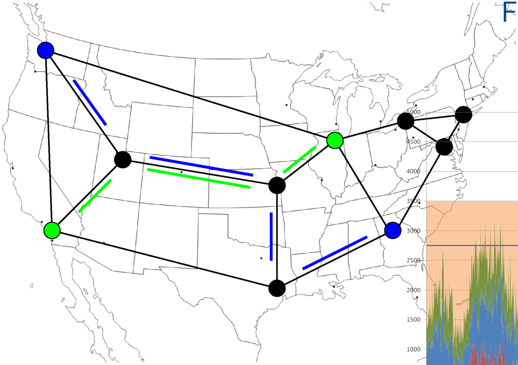
Flaw of using nominal values in optimization problems

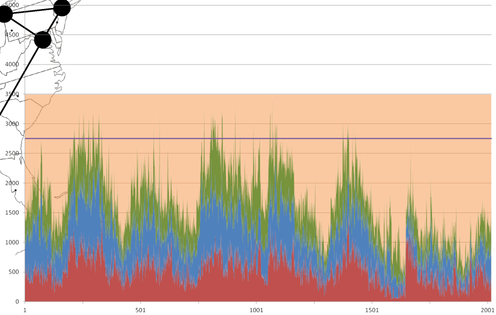
RO: find a solution that is robust against this uncertainty in the parameters.

Slide copied from Dick den Hertog (University of Tilburg)

Robust Design of Networks

Given	network topology potential link capacities uncertain demands
Find	link dimensioning routing at minimum cost





Purple: Sum of 90% quantiles

Lower overestimation

The network is designed such that capacities are as small as possible; traffic fluctuations might result in high network congestion

Stochastic Programming

Network design has to be computed for many scenarios; high computational effort

Multi-period Network Design

Many traffic matrices have to be considered simultaneously; high computational effort

Our choice: Chance-constrained Programming with Robust Optimization as “special case”

Outline

1 Motivation
2 Discrete Optimization Basics
2.1 Polyhedral Combinatorics
2.2 Example
2.3 Chvátal-Gomory Cuts & Mixed Integer Rounding
2.4 ILP Software

Linear Programming Models

- Many optimization problems can be formulated as linear programs:
- Variables $x \in \mathbb{R}^n_+$ model the decisions
- Linear constraints $a_i x \le b_i$ $(i = 1, \dots, m)$ model the relations between the decisions (conflicts, conclusions, etc.)
- The best decision can be found by minimizing/maximizing an objective $\sum_{j=1}^n c_j x_j$
- The linear program reads:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \le b \\ & x \ge 0 \end{array}$$

- or $\min_{x \in P} c^T x$ with $P = \{x \in \mathbb{R}^n_+ : Ax \le b\}$

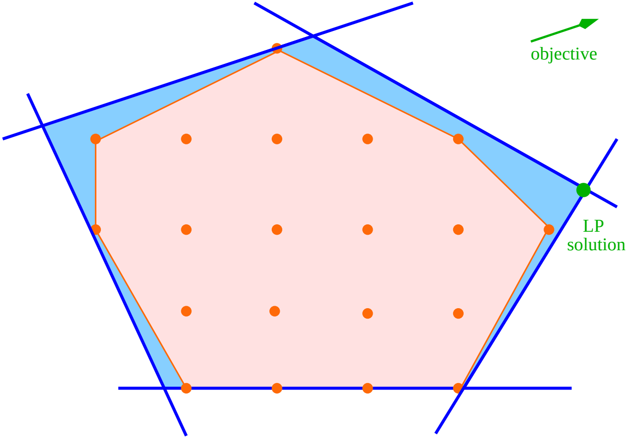
Integer Linear Programming Models

- Often decisions have a discrete nature:
  - On/Off
  - Install $0, 1, 2, \ldots$ units of capacity
  - Select channel $1, 2, \ldots, 12$ or 13 for transmission by a WLAN access point
- Variables $x_j \in \mathbb{Z}_+$ or $x_j \in \{0, 1\}$ model the decisions
- The integer linear program reads:

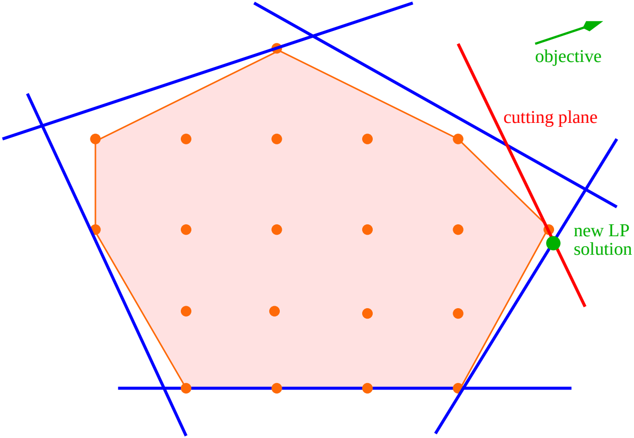
$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{Z}_+^n \end{aligned}$$

- or $\min_{x \in P \cap \mathbb{Z}^n} c^T x$ with $P = \{x \in \mathbb{R}_+^n : Ax \leq b\}$

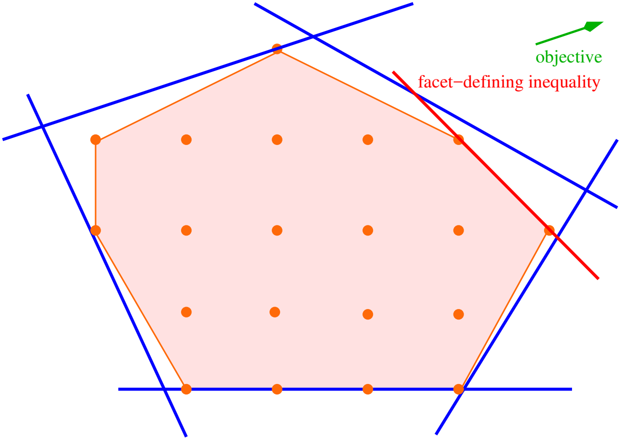
Integer Linear Programming



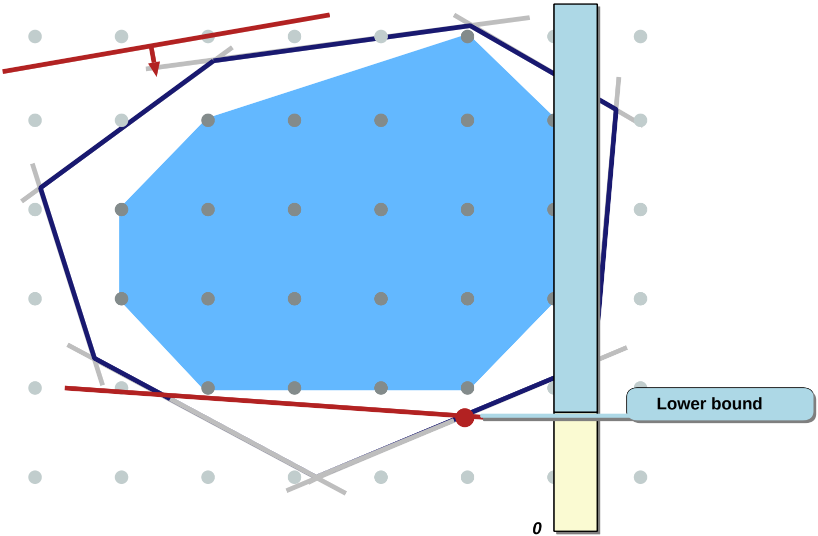
- Linear programs can be solved efficiently (in theory and practise)
- (Mixed) Integer linear programs are harder to tackle
- Knowledge about convex hull of integer solutions is needed



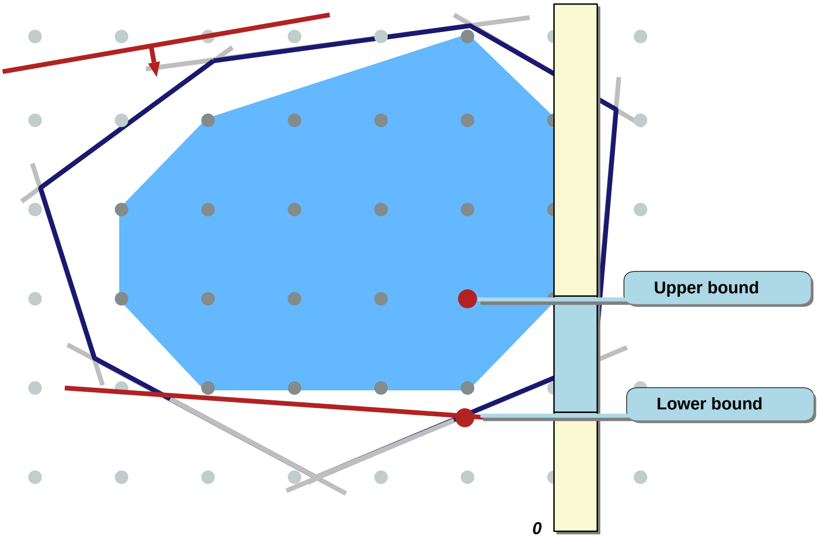
- Solution of LP relaxation is not part of the convex hull
- Explore problem structure: valid inequalities
- Add violated inequality to LP relaxation: cutting plane



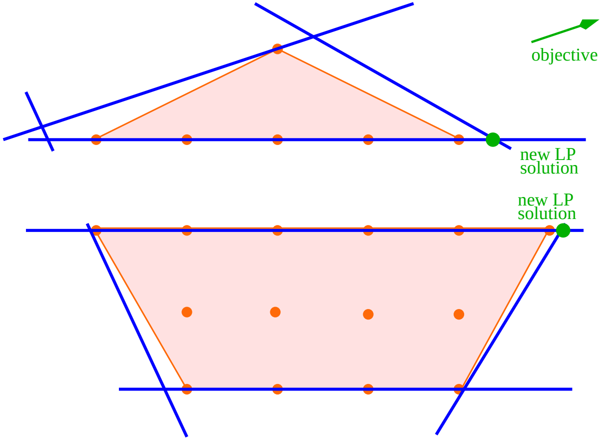
- Strong valid inequalities are problem dependent
- A facet is a face of the convex hull of integer points
- Strongest valid inequalities define facets
- If enough facets are known, problem can be solved as LP



Lower bound improves by adding cutting planes

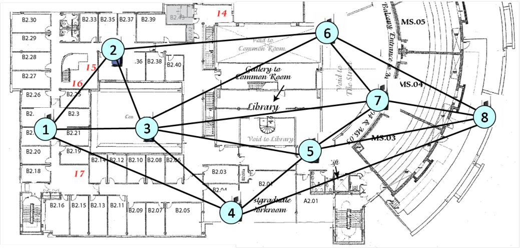


Upper bound can be computed via heuristic or Branch & Bound



- Fractional LP solution and no violated inequalities found
- Branch on fractional value of integer variable
- Bound solution space by best solution
- Cutting plane + Branch & Bound: Branch & Cut

WLAN Frequency Planning



- Warwick Mathematics Institute
- WLAN Access Points
- Edges represent interference if APs operate on same channel
- How many APs can be operated at the same channel/frequency?

Polyhedral combinatorics – Example

Definition Let $G = (V, E)$ be a graph. A stable set S is a subset of the nodes such that no for all $v, w \in S$, $\{v, w\} \notin E$.

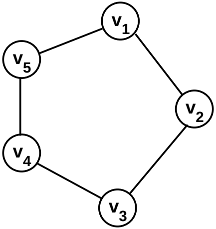
$$z_{IP} = \max \sum_{v \in V} x_v \tag{1}$$

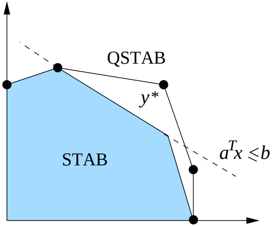
$$s.t.\ x_v + x_w \leq 1 \qquad \forall \{v, w\} \in E \tag{2}$$

$$x_v \in \{0, 1\} \qquad \forall v \in V \tag{3}$$

Notation:

- $STAB(G) := \text{conv}\{x \in \{0,1\}^n : x_v + x_w \leq 1\ \forall \{v, w\} \in E\}$
- $ESTAB(G) := \{x \in [0,1]^n : x_v + x_w \leq 1\ \forall \{v, w\} \in E\}$
- $z_{IP} = \max\{\sum_{v \in V} x_v : x \in STAB(G)\}$
- $z_{LP} = \max\{\sum_{v \in V} x_v : x \in ESTAB(G)\}$





C_5: $z_{IP} = 2$, $z_{LP} = 2\frac{1}{2}$ ($x_v = \frac{1}{2}$)

How does $ax \leq b$ look like?

Odd cycle inequality: $\sum_{v \in C_5} x_v \leq 2$

Validity: At most 2 nodes can be in any stable set

$$\max \left\{ \sum_{v \in C_5} x_v : x \in ESTAB(C_5), \sum_{v \in C_5} x_v \leq 2 \right\} = 2$$

Strength of valid inequalities

Dimension of a polytope: number of linearly independent directions

Lemma

$\dim(STAB(G)) = n$ (number of nodes)

Facet of a polytope P: face of dimension $\dim(P) - 1$

Observation

A polytope (the convex hull of integer points) is completely described by its facet-defining inequalities $a_i x \leq b_i$ for $i = 1, \ldots, N$ (N can be very large)

Lemma

- $\sum_{v \in C_5} x_v \leq 2$ describes a facet of $STAB(C_5)$
- $\sum_{v \in C_{2k+1}} x_v \leq k$ describes a facet of $STAB(C_{2k+1})$ for $k \in \{1, 2, \ldots\}$

Lemma

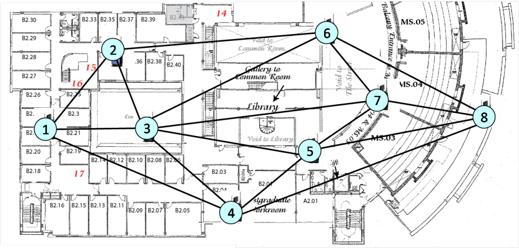
- $\sum_{v \in C_{2k+1}} x_v \leq k$ describes a facet of $STAB(C_{2k+1})$ for $k \in \{1, 2, \ldots\}$

Proof: There exist $2k+1$ affinely independent solutions with $\sum_{v \in C_{2k+1}} x_v = k$ (characterized by $x_{v_i} = x_{v_{i+1}} = 0$ for some $i = 1, \ldots, 2k+1$).

Observation

Let S induce a subgraph of G and let $ax \leq b$ be a facet-defining inequality for $STAB(G[S])$. Then, $ax \leq b$ is a strong *but* not necessarily facet-defining inequality for $STAB(G)$.

WLAN Frequency Planning



- Which APs are operated at which channel/frequency?
- Which AP locations have to be selected?

Another view on odd cycle inequalities:

Inequalities defining $ESTAB(C_5)$

$$
\begin{array}{rrrrrl}
x_1 & +x_2 & & & & \leq 1 \\
 & x_2 & +x_3 & & & \leq 1 \\
 & & x_3 & +x_4 & & \leq 1 \\
 & & & x_4 & +x_5 & \leq 1 \\
x_1 & & & & +x_5 & \leq 1
\end{array}
$$

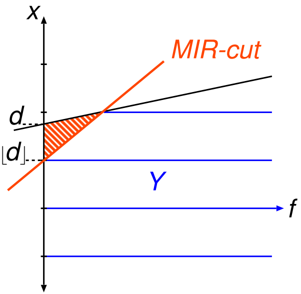
$$
\text{sum:} \tfrac{1}{2} * (\quad 2x_1 \quad +2x_2 \quad +2x_3 \quad +2x_4 \quad +2x_5 \leq 5) \quad x_1 \quad + x_2 \quad + x_3 \quad + x_4
$$

Chvátal-Gomory cuts

Given a system of linear inequalities $Ax \leq b$ and a vector μ, the Chvátal-Gomory cut $\lfloor \mu^T A \rfloor x \leq \lfloor \mu^T b \rfloor$ is valid for all integer solutions of the system.

$\Rightarrow$ Chvátal-Gomory cuts are among the most powerful general cutting planes applied in state-of-the-art Branch & Cut solvers like ILOG CPLEX, GuRoBi, XPRESS MP, or SCIP.

Beyond Chvátal-Gomory cuts:



Base inequality: $af + x \leq d$

where $f \in \mathbb{R}_+,\ x \in \mathbb{Z}_+$

MIR inequality: $\frac{a}{1-<d>} f + x \leq \lfloor d \rfloor$,

where $< d >= d - \lfloor d \rfloor$

Note: MIR introduces integral vertices!

Base:

$$\sum_{i=1}^{m} a_j f_j + \sum_{j=1}^{n} c_j x_j \geq d$$

Apply MIR function:

$$\sum_{i=1}^{m} \bar{F}_{d,c}(a_j) f_j + \sum_{j=1}^{n} F_{d,c}(c_j) x_j \geq F_{d,c}(d)$$

where

$$\begin{aligned} F_{d,c}(a) &:= r(d,c)\lceil a \rceil - (r(d,c) - r(a,c))^{+} \\ \bar{F}_{d,c}(a) &:= r(d,c) a^{+} = \lim_{t \searrow 0} F_{d,c}(at)/t \\ r(d,c) &:= a - c\left(\left\lceil \frac{a}{c} \right\rceil - 1\right) \end{aligned}$$

Details: $F_{d,c}$ has to be subadditive and nondecreasing with $F_{d,c}(0) = 0$.
(see Nemhauser/Wolsey 1988)

Examples:

- ZIMPL: modelling only, opensource, `http://zibopt.zib.de`
- Example of ZIMPL:

```
# This is the stable set problem for the Petersen graph
set V := 1 .. 10 ;
set E := <1,2>, <1,5>, <1,6>, <2,3>, <2,7>,
<3,4>, <3,8>, <4,5>, <4,9>, <5,10>,
<6,8>, <6,9>, <7,9>, <7,10>, <8,10> ;
var x[V] binary;
maximize stableset: sum <v> in V : x[v];
subto conflict:
forall <v,w> in E do
x[v] + x[w] <= 1;
# That's it.
```

Examples:

- ZIMPL: modelling only, opensource, `http://zibopt.zib.de`
- AIMMS: modelling + black-box solver, commercial, `http://www.aimms.com`
- AMPL, ILOG OPL Studio, commercial, `http://www-01.ibm.com/software/websphere/products/optimization/`
- GAMS, `http://www.gams.com`

Advantages: Perfect for testing of models and black-box ILP solving.
Disadvantages: Limited interaction to enhance solution process.

Examples:

- IBM ILOG CPLEX: commercial, http://www-01.ibm.com/software/websphere/products/optimization/
- GuRoBi: commercial, but free for academic usage, http://www.gurobi.com
- SCIP: opensource, http://zibopt.zib.de
- COIN-OR: opensource, http://www.coin-or.org
- LP solvers: SOPLEX, CLP, ...

Advantages: Full control of many parameter settings, callback functionality for user defined cutting planes, heuristics, etc.
Disadvantages: Full control only via C/C++/JAVA/python interface; careful programming required to avoid invalid inequalities, etc.