Algorithmic Graph Theory: How hard is your combinatorial optimization problem?

Arie M.C.A. Koster

Lecture 3

Clemson, June 8, 2017

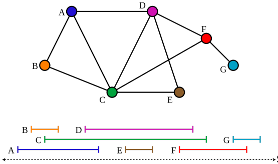
CLEMSON UNIVERSITY
Lehrstuhl II für Mathematik
RWTH AACHEN UNIVERSITY

Definition

The intersection graph of a collection $\mathcal{F}$ of non-empty sets contains a vertex for every set $F \in \mathcal{F}$ and an edge $\{v, w\}$ if and only if the two corresponding sets intersect.

Definition

The intersection graph of a collection $\mathcal{F}$ of non-empty sets contains a vertex for every set $F \in \mathcal{F}$ and an edge $\{v, w\}$ if and only if the two corresponding sets intersect.



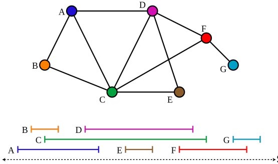
source: wikipedia

Definition

A interval graph is an intersection graph of a collection of intervals on the real line.

Definition

The intersection graph of a collection $\mathcal{F}$ of non-empty sets contains a vertex for every set $F \in \mathcal{F}$ and an edge $\{v, w\}$ if and only if the two corresponding sets intersect.



source: wikipedia

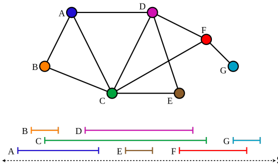
Definition

A interval graph is an intersection graph of a collection of intervals on the real line.

- unit interval graph: $|I_p| = 1$ for all $p \in \{1, \ldots, n\}$

Definition

The intersection graph of a collection $\mathcal{F}$ of non-empty sets contains a vertex for every set $F \in \mathcal{F}$ and an edge $\{v, w\}$ if and only if the two corresponding sets intersect.



source: wikipedia

Definition

A interval graph is an intersection graph of a collection of intervals on the real line.

- unit interval graph: $|I_p| = 1$ for all $p \in \{1, \ldots, n\}$
- proper interval graph: $I_p \not\subset I_q$ for all $p, q \in \{1, \ldots, n\}$

Lemma

An induced subgraph of an interval graph is again an interval graph.

Lemma

An induced subgraph of an interval graph is again an interval graph.

Lemma

An interval graph does not contain induced cycles of length 4 or more.

Lemma

An induced subgraph of an interval graph is again an interval graph.

Lemma

An interval graph does not contain induced cycles of length 4 or more.

Definition

A graph is called chordal (or triangulated) if it does not contain induced cycles with length 4 or more, i.e., every cycle of length ≥ 4 contain a chord.

Lemma

An induced subgraph of an interval graph is again an interval graph.

Lemma

An interval graph does not contain induced cycles of length 4 or more.

Definition

A graph is called chordal (or triangulated) if it does not contain induced cycles with length 4 or more, i.e., every cycle of length ≥ 4 contain a chord.

Not every chordal graph is an interval graph.

Theorem (Gilmore & Hoffman, 1964)

A graph G is an interval graph if and only if the maximal cliques of G can be ordered linearly, such that for all $v \in V(G)$, the maximal cliques containing v appear consecutively.

Interval Graphs

Theorem (Gilmore & Hoffman, 1964)

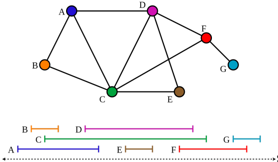
A graph G is an interval graph if and only if the maximal cliques of G can be ordered linearly, such that for all $v \in V(G)$, the maximal cliques containing v appear consecutively.

The clique matrix $M(G)$ contains n rows and m columns (where m is the number of maximal cliques), with

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \in Q_j \\ 0 & \text{otherwise} \end{cases}.$$

Corollary

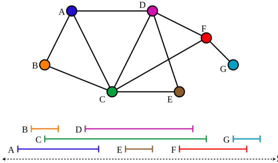
The maximum (weighted) independent set problem can be solved in polynomial time on interval graphs.



source: wikipedia

Theorem

The maximum independent set problem can be solved in $O(n \log n)$ if G is interval.



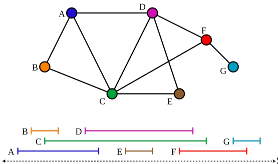
source: wikipedia

Theorem

The maximum independent set problem can be solved in $O(n \log n)$ if G is interval.

Theorem

The maximum clique problem can be solved in $O(n \log n)$ if G is interval.



source: wikipedia

Theorem

The maximum independent set problem can be solved in $O(n \log n)$ if G is interval.

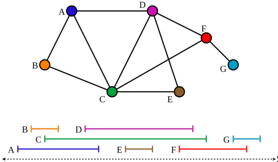
Theorem

The maximum clique problem can be solved in $O(n \log n)$ if G is interval.

Corollary

The maximum weighted clique problem can be solved in $O(n \log n)$ if G is interval.

Interval Graphs



source: wikipedia

$\omega(G)$ = clique number, $\chi(G)$ = chromatic number

Theorem

For interval graphs G it holds $\chi(G) = \omega(G)$ and can be computed in $O(n \log n)$.

Definition

A vertex $v \in V(G)$ is called simplicial if the neighbors $N(v)$ induce a clique in G (and thus also $N[v]$ induces a clique).

Simplicial Vertices

Definition

A vertex $v \in V(G)$ is called simplicial if the neighbors $N(v)$ induce a clique in G (and thus also $N[v]$ induces a clique).

Definition

A subset $S \subseteq V$ is a vertex separator if there exist two non-adjacent vertices a and b such that a and b are in different components of $G[V \setminus S]$. (S is an a-b-Separator.)

Definition

A vertex $v \in V(G)$ is called simplicial if the neighbors $N(v)$ induce a clique in G (and thus also $N[v]$ induces a clique).

Definition

A subset $S \subseteq V$ is a vertex separator if there exist two non-adjacent vertices a and b such that a and b are in different components of $G[V \setminus S]$. (S is an a-b-Separator.)

Theorem

Every chordal graph has a simplicial vertex. If G is not complete, there exist at least two non-adjacent simplicial vertices.

Definition

A vertex $v \in V(G)$ is called simplicial if the neighbors $N(v)$ induce a clique in G (and thus also $N[v]$ induces a clique).

Definition

A subset $S \subseteq V$ is a vertex separator if there exist two non-adjacent vertices a and b such that a and b are in different components of $G[V \setminus S]$. (S is an a-b-Separator.)

Theorem

Every chordal graph has a simplicial vertex. If G is not complete, there exist at least two non-adjacent simplicial vertices.

Corollary

The minimal a-b-separators of a chordal graph induce cliques.

Theorem

A graph is chrodal if and only if every minimal vertex separator induces a clique.

Theorem

A graph is chrodal if and only if every minimal vertex separator induces a clique.

Definition

Let $G = (V, E)$ be a graph and $\sigma = [v_1, \ldots, v_n]$ be an ordering of the vertices. The ordering is called a perfect elimination scheme (PES) if for all $i = 1, \ldots, n$ the vertex v_i is simplicial in $G[v_i, \ldots, v_n]$.

Theorem

A graph is chrodal if and only if every minimal vertex separator induces a clique.

Definition

Let $G = (V, E)$ be a graph and $\sigma = [v_1, \ldots, v_n]$ be an ordering of the vertices. The ordering is called a perfect elimination scheme (PES) if for all $i = 1, \ldots, n$ the vertex v_i is simplicial in $G[v_i, \ldots, v_n]$.

Theorem

A graph G is chordal if and only if a PES exists. Moreover, the PES can start with any simplicial vertex of G.

How to determine if G is chordal?

Lemma

If G is chordal, there exists a PES with an arbitrary vertex as v_n.

How to determine if G is chordal?

Lemma

If G is chordal, there exists a PES with an arbitrary vertex as v_n.

Lexicographic Breadth First Search (LBFS)
Maximum Cardinality Search (MCS)

How to determine if G is chordal?

Lemma

If G is chordal, there exists a PES with an arbitrary vertex as v_n.

Lexicographic Breadth First Search (LBFS)
Maximum Cardinality Search (MCS)

Theorem

A chordal graph with n vertices has at most n maximal cliques (with equality if and only if no edges exist).

How to determine if G is chordal?

Lemma

If G is chordal, there exists a PES with an arbitrary vertex as v_n.

Lexicographic Breadth First Search (LBFS)
Maximum Cardinality Search (MCS)

Theorem

A chordal graph with n vertices has at most n maximal cliques (with equality if and only if no edges exist).

Theorem

For a chordal graph it holds that $\chi(G) = \omega(G)$.

Let

$$\begin{aligned} y_1 &= \sigma(1) \\ y_i &= \sigma\left(\min\{j \leq n : \sigma(j) \notin X_{y_1} \cup X_{y_2} \cup \cdots \cup X_{y_{i-1}}\}\right) \end{aligned}$$

until no further vertices exist.. In the end, there exists a $t > 0$ such that

$$\{y_1, y_2, \ldots, y_t\} \cup X_{y_1} \cup X_{y_2} \cup \cdots \cup X_{y_t} = V.$$

Theorem

The set $\{y_1, \ldots, y_t\}$ is a maximum independent set.

Let

$$y_1 = \sigma(1)$$
$$y_i = \sigma\left(\min\{j \leq n : \sigma(j) \notin X_{y_1} \cup X_{y_2} \cup \cdots \cup X_{y_{i-1}}\}\right)$$

until no further vertices exist.. In the end, there exists a $t > 0$ such that

$$\{y_1, y_2, \ldots, y_t\} \cup X_{y_1} \cup X_{y_2} \cup \cdots \cup X_{y_t} = V.$$

Theorem

The set $\{y_1, \ldots, y_t\}$ is a maximum independent set.

Corollary

For chordal graphs, $k(G) = \alpha(G)$ with $k(G)$ the clique cover number, i.e., the minimum number of cliques to cover all vertices of G.

Let

$$\begin{aligned} y_1 &= \sigma(1) \\ y_i &= \sigma\left(\min\{j \leq n : \sigma(j) \notin X_{y_1} \cup X_{y_2} \cup \cdots \cup X_{y_{i-1}}\}\right) \end{aligned}$$

until no further vertices exist.. In the end, there exists a $t > 0$ such that

$$\{y_1, y_2, \ldots, y_t\} \cup X_{y_1} \cup X_{y_2} \cup \cdots \cup X_{y_t} = V.$$

Theorem

The set $\{y_1, \ldots, y_t\}$ is a maximum independent set.

Corollary

For chordal graphs, $k(G) = \alpha(G)$ with $k(G)$ the clique cover number, i.e., the minimum number of cliques to cover all vertices of G.

Corollary

Chordal graphs are perfect.

Definition

A collection $\{T_i\}_{i \in I}$ of subsets of a set T has the Helly property if $J \subset I$ with $T_i \cap T_j \neq \emptyset$ for all $i, j \in J$ implies: $\bigcap_{j \in J} T_j \neq \emptyset$.

Definition

A collection $\{T_i\}_{i \in I}$ of subsets of a set T has the Helly property if $J \subset I$ with $T_i \cap T_j \neq \emptyset$ for all $i, j \in J$ implies: $\bigcap_{j \in J} T_j \neq \emptyset$.

Lemma

Let T be a tree and T_i a subtree of T for all $i \in I$. Then, the collection of subtrees has the Helly property.

Definition

A collection $\{T_i\}_{i \in I}$ of subsets of a set T has the Helly property if $J \subset I$ with $T_i \cap T_j \neq \emptyset$ for all $i, j \in J$ implies: $\bigcap_{j \in J} T_j \neq \emptyset$.

Lemma

Let T be a tree and T_i a subtree of T for all $i \in I$. Then, the collection of subtrees has the Helly property.

Theorem

Let G be a graph. The following properties are equivalent:

1. *G is chordal*
2. *G is the intersection graph of a collection of subtrees of a tree*
3. *there exists a tree $T = (K, L)$ such that node set K represents all maximal cliques in G and edge set L is chosen such that the subgraph induced by $K_v := \{Q \in K : v \in Q \text{ clique in } G\}$ represents a subtree.*

Algorithmic Graph Theory: How hard is your combinatorial optimization problem?

Arie M.C.A. Koster

Lecture 3

Clemson, June 8, 2017

Lehrstuhl II für Mathematik