Algorithmic Graph Theory: How hard is your combinatorial optimization problem?

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Lecture 2

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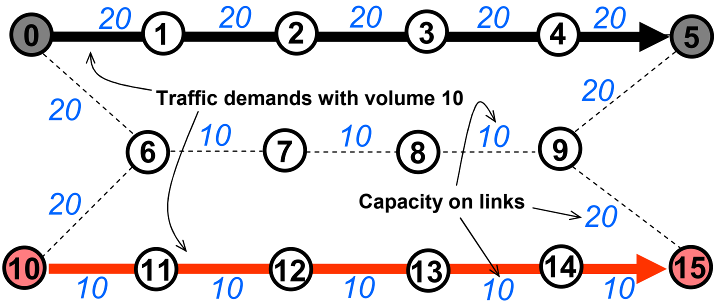
Network Design with Compression:

- Given a Network $G = (V, E)$,
- With capacity $c_{uv} = c \geq 0$ for all edges uv.
- 2 Demands d^1, d^2 (with a potential compression rate λ).



Figures provided by Truong Khoa Phan

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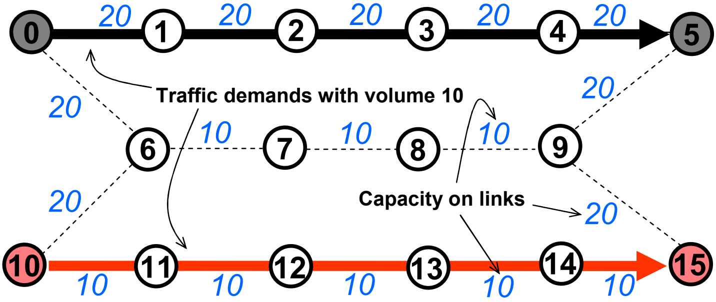

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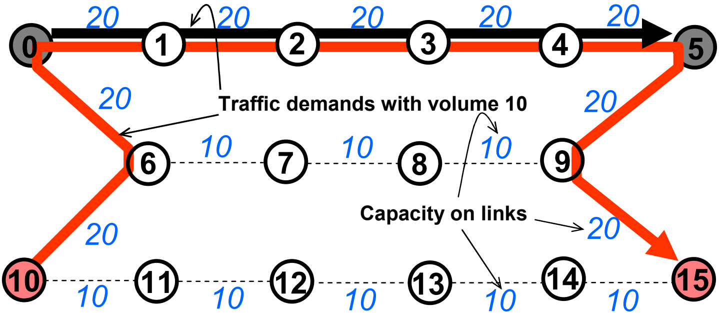

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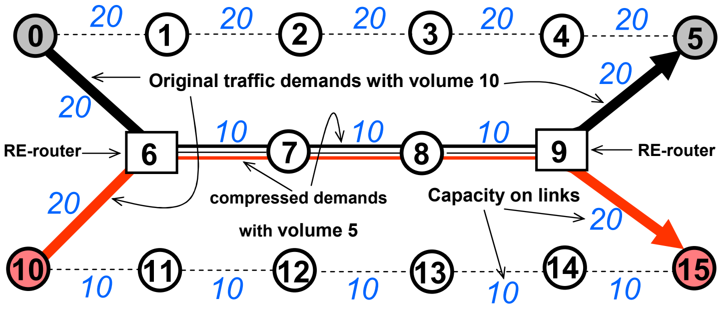
- Employ Compression if beneficial.



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- Variables:

$f_{vu}^{st} \in \mathbb{R}_{\geq 0}$:	Fraction of demand st routed uncompressed on edge vu.
$g_{vu}^{st} \in \mathbb{R}_{\geq 0}$:	Fraction of demand st routed compressed on edge vu.
$x_{uv} \in \mathbb{Z}_{\geq 0}$:	Usage of edge uv.
$y_v \in \{0, 1\}$:	Whether compression enabled at node v.

NDPC as MILP

$$\min \sum_{uv \in E} C_{uv} x_{uv} + \sum_{v \in V} C_v y_v$$

$$\text{s.t.} \sum_{u \in N(v)} \left(f^q_{vu} + g^q_{vu} - f^q_{uv} - g^q_{uv}\right) = \begin{cases} -1 & \text{if } u = s^q, \\ 1 & \text{if } u = t^q, \\ 0 & \text{else} \end{cases} \qquad \forall\, v \in V, \forall\, q \in Q$$

$$\sum_{q \in Q} \left(d^q \left(f^q_{uv} + f^q_{vu}\right) + \lambda d^q \left(g^q_{uv} + g^q_{vu}\right)\right) \leq c x_{uv} \qquad \forall\, uv \in E$$

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Compression vs. No Compression

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$\|V\|$	12	17
$\|E\|$	15	26
$\|Q\|$	130	251

CPU time to optimality

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CPU time to optimality

Conclusion: Complexity increases significantly!

Cost of links: C_{uv}
Objective: $\min \sum_{uv \in E} C_{uv} x_{uv}$

Network Design w/o Compression

The NETWORK DESIGN problem is NP-hard.

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The Network Design problem is NP-hard.

Cost of compression at node v: C_v
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Network Design with Compression

Corollary: Network Design with Compression is NP-hard as well.

Complexity of the General Problem

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Network Design with Compression

Corollary: NETWORK DESIGN WITH COMPRESSION is NP-hard as well.

but, is it "more difficult" than NETWORK DESIGN?

Easy case (w/o Compression!):

If G is a tree, Network Design can be solved in polynomial time.

Network Design on a Tree

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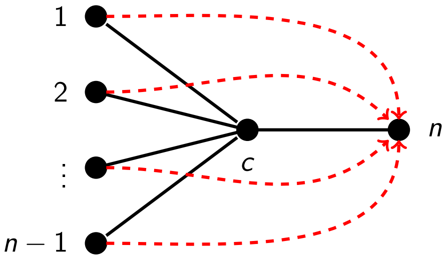
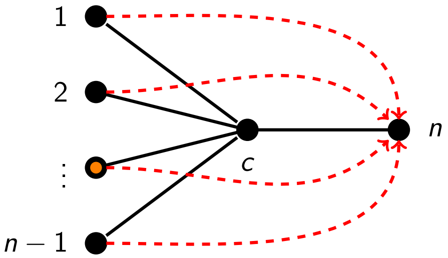
Definition

Given a fixed routing, the **compressor placement problem** is to determine the active compressors and link capacities at minimum cost.



- Star G with $n+1$ vertices
- Demands d_{in}, $i = 1, \ldots, n-1$
- Capacity of c on every link (very expensive to install more)



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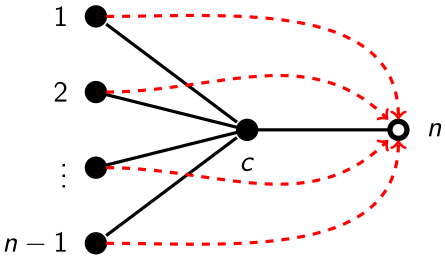

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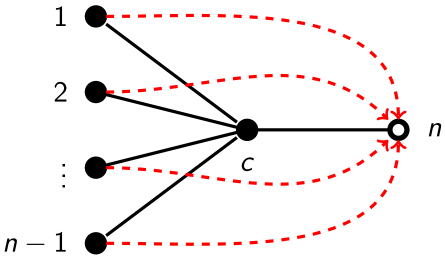

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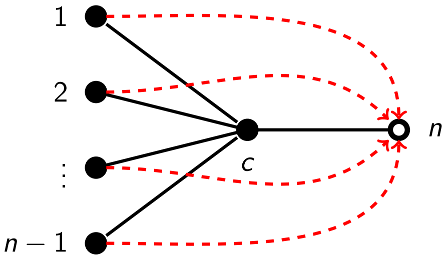

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  Where to place converters? In center-node, or individual nodes $1, \dots, n-1$?

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Proof:

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Proof: Reduction from Knapsack

- profits c_i, $i \in N := \{1, \ldots, n-1\}$
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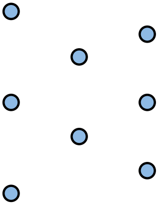
Reduction from HITTING SET: "Universe" U, subsets $S_i \subseteq U$, and integer k, $\exists H \subseteq U$ with $|H| \leq k$ such that $H \cap S_i \neq \emptyset \ \forall i = 1, \ldots, n$?



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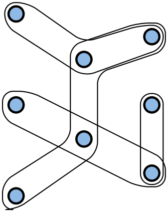

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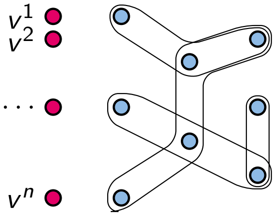

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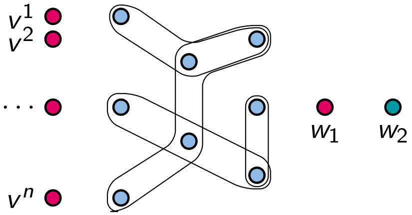

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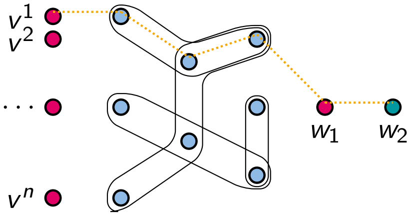

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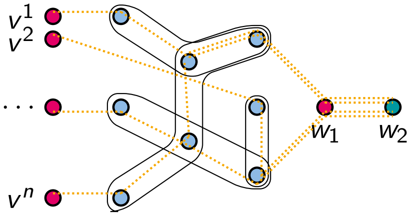
Reduction from HITTING SET: "Universe" U, subsets $S_i \subseteq U$, and integer k, $\exists H \subseteq U$ with $|H| \leq k$ such that $H \cap S_i \neq \emptyset \ \forall i = 1, \ldots, n$?



Idea suggested by Stéphane Perennes

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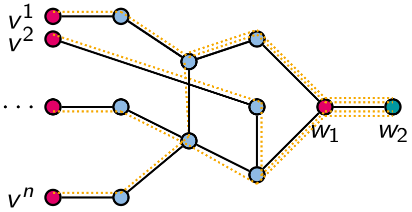

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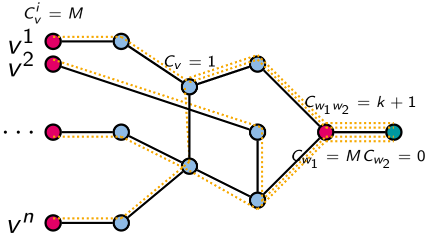

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demand $d_{v^i w_2} = \frac{2}{n}$

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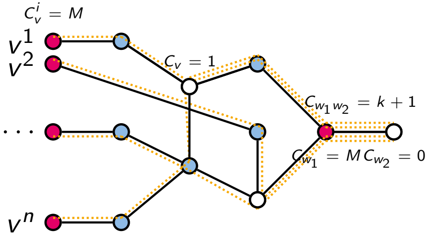

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$|H| \leq k$ if and only if Cost of NDPC $\leq 2k + 1$

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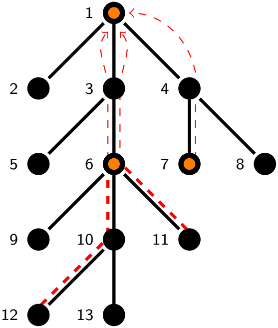
Network Design with Compression *on trees is weakly NP-hard.*

- Tree $G = (V, E)$, $|V| = n$, nodes labeled increasingly by BFS, starting at $i = 1$



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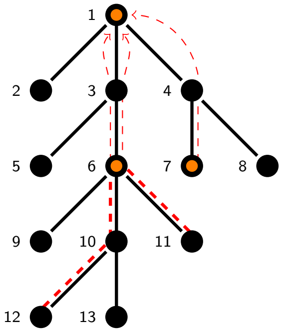
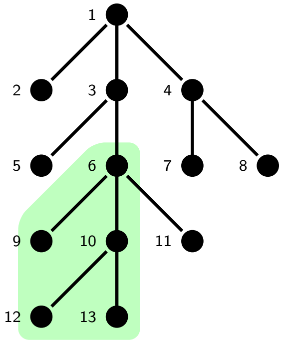
- Tree $G = (V, E)$, $|V| = n$, nodes labeled increasingly by BFS, starting at $i = 1$
- Capacity $c \in \mathbb{Z}_{\geq 0}$ per installed batch
- Commodities $Q = \{(i, 1) : i \geq 2\}$, $d_i \in \mathbb{Z}_{\geq 0}$, direct routing



Notation

- $[i,k]$ subtree induced by i and offspring of i's first k children



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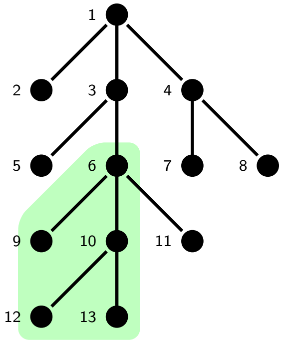

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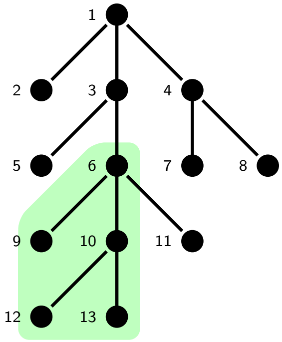

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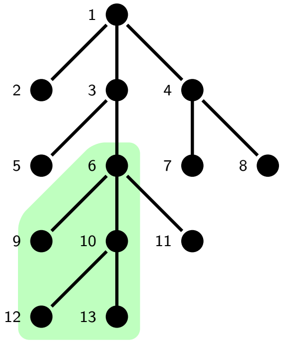
- $d([i, k])$ demand induced by subgraph $[i, k]$



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Cost functions:

- $\mathcal{C}([i,k],f)$: min cost of $[i,k]$ with **compressing** in i and uncompressed flow of f on $(i,p(i))$ (but cost not counted yet)
- $\mathcal{D}([i,k],f)$: min cost of $[i,k]$ with **decompressing** in i and uncompressed flow of f on $(i,p(i))$
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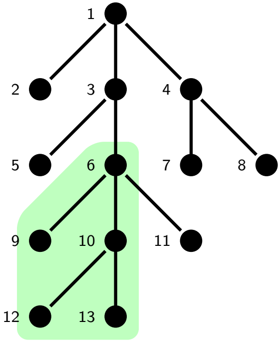

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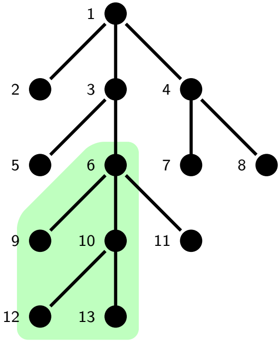

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Lemma

Given a tree instance, an optimal solution of NDPC *is given by*
$\min\{\mathcal{D}([1,a(1)],0),\mathcal{N}([1,a(1)],0)\}$.

Observation: For $i \neq 1$, only $\mathcal{C}([i,k], 0)$ and $\mathcal{D}([i,k], d[i,k])$ needed.

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Lemma (Initialization)

For $i \in V \setminus \{1\}$ and $f \in \mathbb{Z}_+$, it is

- $\mathcal{C}([i,0],0) = C_i$,
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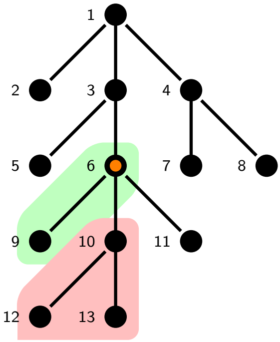

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Network Design on a Tree

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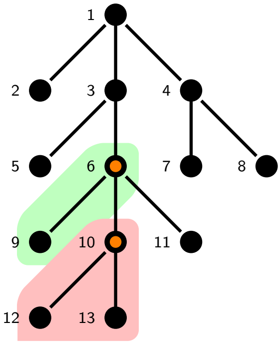

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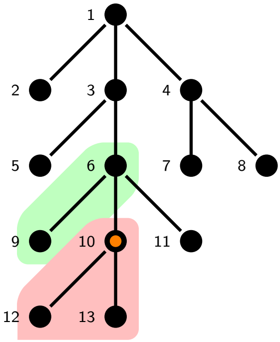

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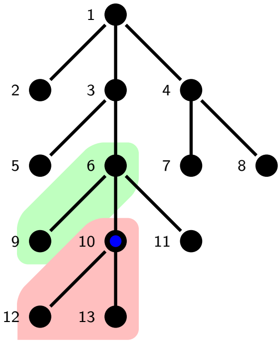

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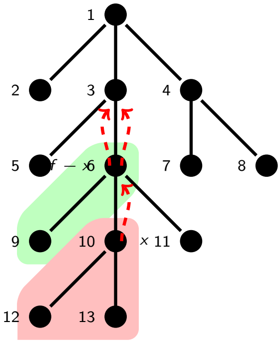

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NDPC *on trees can be solved in* $O(n^3 \triangle^2)$.

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- Total runtime of $O(n^3 \triangle^2)$

Train Packing Problem

Given a set of commodities $Q = \{(s^q, t^q, d^q) : q = 1, \ldots, |Q|\}$, a network $G = (V, A)$, a set of shunting yards $R \subset V$, and a train capacity C, determine the minimum number of trains needed to transport all demands, where each train can be rearranged at shunting yards R (but at least one commodity should continue with the same train).

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A Rail Cargo Problem

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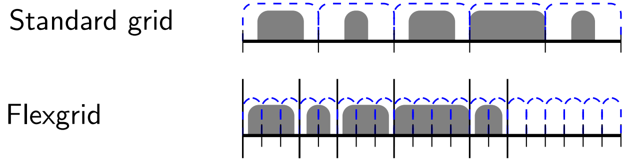
$$\text{number of trains} = \begin{cases} \frac{(n-1)(n+1)}{8} & \text{if } n = 2k+1 \\ \frac{n^2}{8} & \text{if } n = 4k \\ \frac{n^2}{8} + \frac{1}{2} & \text{if } n = 4k+2 \end{cases}$$

Idea: fixed spectrum-block size → flexible block-size



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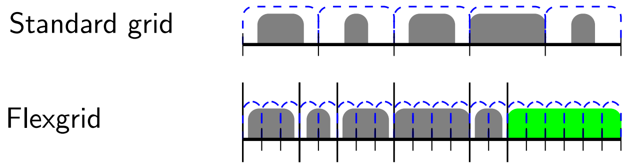

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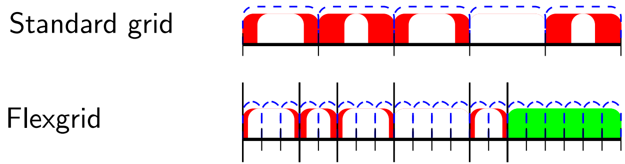
- Demands request a custom amount of these slots ('size')

Idea: fixed spectrum-block size → flexible block-size



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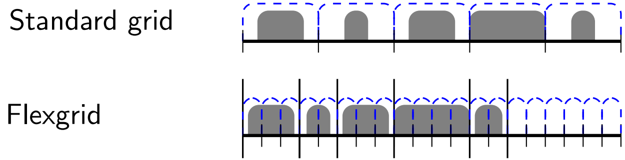

- Spectrum is divided into a small slots (e.g. 6.25GHz)
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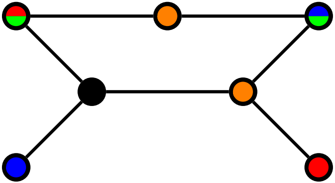
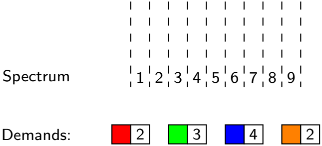
- "Freedom" is paid for: contiguity of assigned slots required



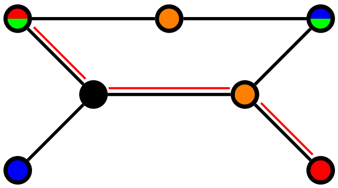


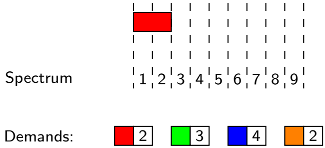
Definition (*Spectrum Allocation Problem* (SA))

Given a simple undirected graph $G = (V, E)$ and a set R of pairs $R_i = (P_i, d_i) \in \mathcal{P} \times \mathbb{N}$, $1 \leq i \leq l$, determine

1. for every R_i an interval $I_i = [a_i, b_i)$ with $a_i \leq b_i \in \mathbb{N}$ und $b_i - a_i = d_i$, such that $\max\{b_i | i = 1, \dots, l\}$ minimal, where $I_i \cap I_j = \emptyset$ if paths P_i and P_j share an edge in G.

Let $SA(G, R)$ denote the value of an optimal solution.





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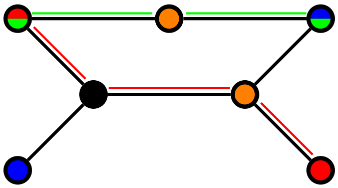

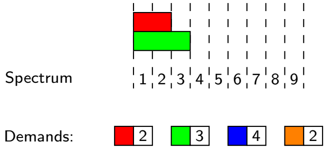

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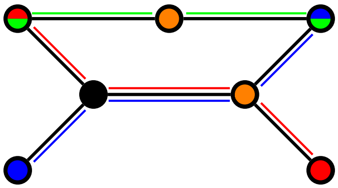
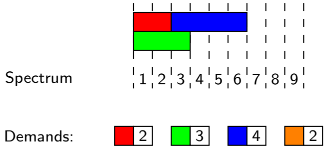
Spectrum Allocation Problem





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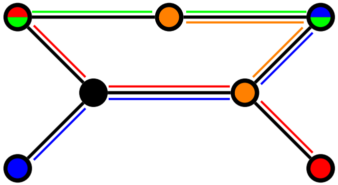

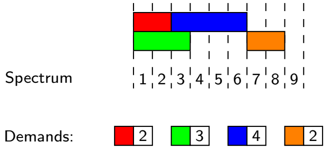

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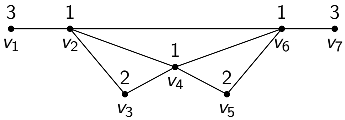
Definition (Interval-Coloring Problem (IC))

Let $G = (V, E)$ and $d : V \mapsto \mathbb{N}$. The *Interval Coloring Problem* is to assign to every vertex v an interval of length $d(v)$, such that adjacent vertices are assigned disjoint intervals. $\chi_I(G) =$ the minimum of colors required.



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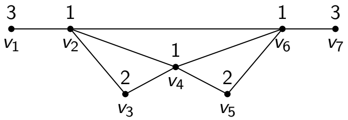

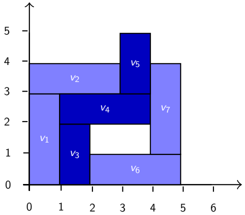

$SA(G, R, P) = \chi_I(G')$

The Spectrum Allocation Problem (G, R, P) is equivalent to the Interval-Coloring Problem on the edge-intersection graph G' of paths P_i.

Corollary

Spectrum Allocation is $\mathcal{NP}$-hard on general networks as well as on star networks

Proof for star networks: wavelength assignment ($d_i = 1$) is $\mathcal{NP}$-hard by a reduction from edge coloring.

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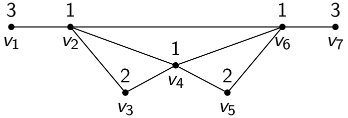
Corollary

Spectrum Allocation is already $\mathcal{NP}$-hard on path networks and $d_i \in \{1, 2\}$

Proof: Interval-Coloring on a path is equivalent to Dynamic Storage Allocation, which is known to be $\mathcal{NP}$-hard.

Definition (Interval-Coloring Problem (IC))

Given a graph $G = (V, E)$ and a weight function $d : V \mapsto \mathbb{N}$, the *Interval Coloring Problem* is to assign to every vertex v an interval of length $d(v)$, such that adjacent vertices are not assigned common colors. Let $\chi_I(G)$ denote the minimum of colors required.



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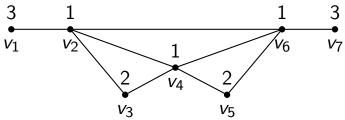

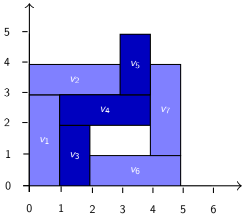

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Definition (Star)

A star $K_{1,n}$ is a graph with vertex set $V(K_{1,n}) = \{v_0, \ldots, v_n\}$ and edge set $E(K_{1,n}) = \{(v_0, v_i) | i = 1, \ldots, n)\}$.

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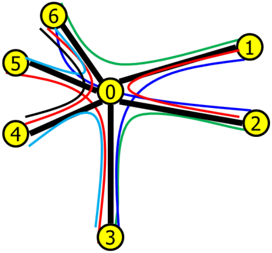
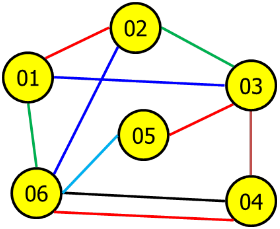
Lemma

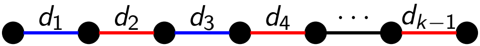
The (R)SA problem on stars is NP-hard, even if all $d_i = 1$.

Proof: Equivalent to EDGE INTERVAL-COLORING on a multigraph



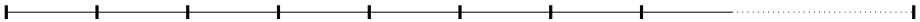


Edge Interval-Coloring on P_k



Color edges with intervals $[a_i, b_i)$ of length d_i such that

$$[a_{i-1}, b_{i-1}) \cap [a_i, b_i) = \emptyset = [a_i, b_i) \cap [a_{i+1}, b_{i+1})$$



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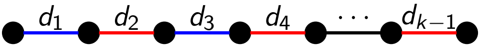

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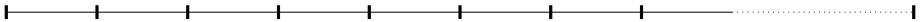
Optimal Solution: Consider odd and even edges separately



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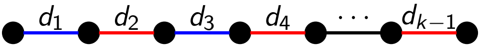

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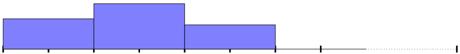
- Assign $[0, d_{2j+1})$ for $j = 0, \ldots, \lfloor \frac{k-1}{2} \rfloor - 1$



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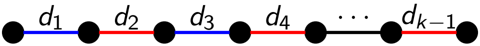

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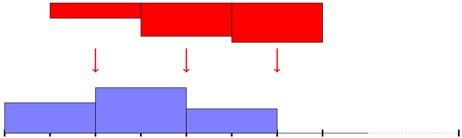
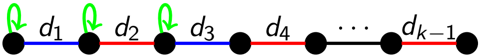
Optimal Solution: Consider odd and even edges separately

- Assign $[0, d_{2j+1})$ for $j = 0, \ldots, \lfloor \frac{k-1}{2} \rfloor - 1$
- Assign $[\chi - d_{2j}, \chi)$ for $j = 1, \ldots, \lfloor \frac{k-1}{2} \rfloor$ with $\chi := \max_{j=1,\ldots,k-2}\{d_j + d_{j+1}\}$



$|P_i| = 1 \Leftrightarrow$ Loop in G'



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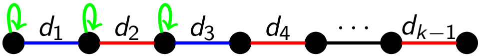

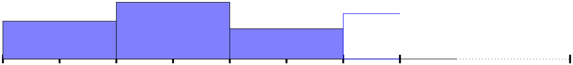

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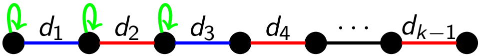

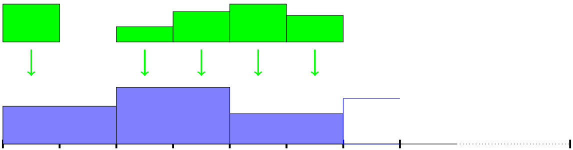

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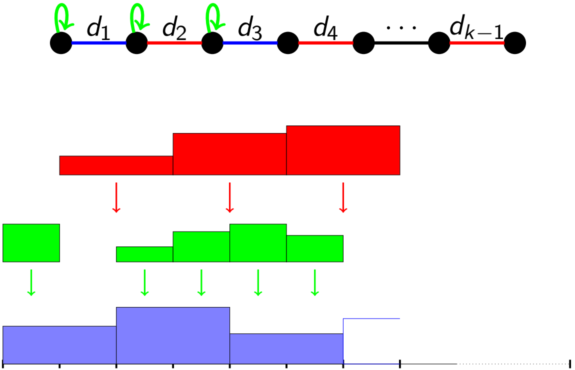
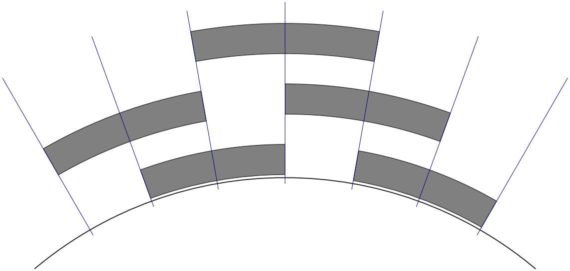


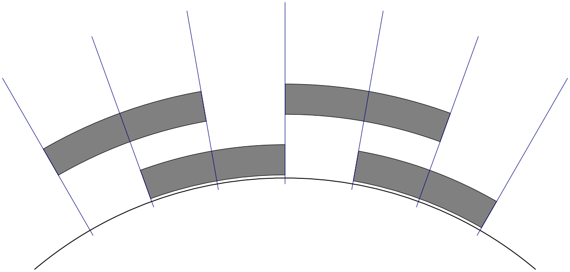
k even: analogue to paths



k even: analogue to paths

k odd:

- after removal of a single edge, we obtain a path



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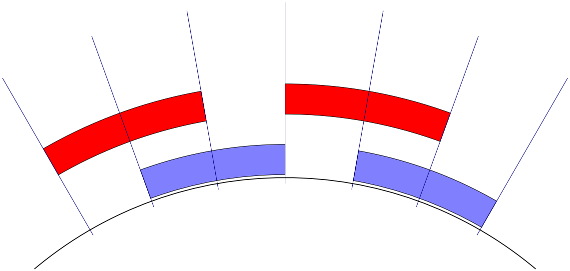

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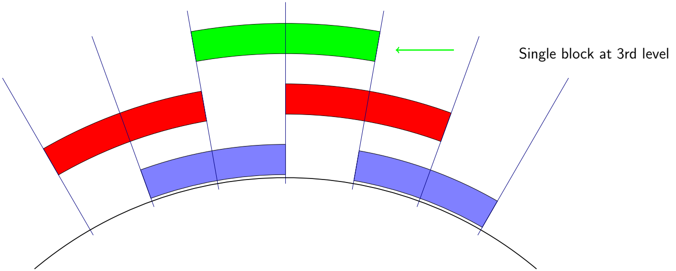
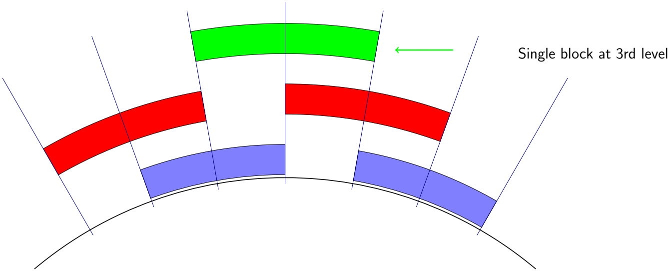


Edge Interval-Coloring on C_k

k even: analogue to paths

k odd:

- after removal of a single edge, we obtain a path
- removed edge requires third level
- search for edge (v_j, v_{j+1}) such that $d_{j-1} + d_j + d_{j+1}$ is minimized



Algorithmic Graph Theory: How hard is your combinatorial optimization problem?

Arie M.C.A. Koster

Lecture 2

Clemson, June 7, 2017

CLEMSON UNIVERSITY
Lehrstuhl II für Mathematik
RWTH AACHEN UNIVERSITY